A graph whose zero-forcing polynomial is not unimodal

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Abstract

The zero-forcing polynomial of a graph counts its zero-forcing sets by cardinality. We give a graph on 73 vertices with 2096, 1592 and 1600 zero-forcing sets of sizes 69, 70 and 71, respectively. This answers in the negative the question of Boyer and coauthors whether the zero-forcing polynomial of every graph is unimodal. The graph joins two disjoint complete graphs, of orders 32 and 33, to four disjoint edges. The proof is a short count: vertices in each clique have the same neighbors outside it, and a zero-forcing set can leave at most one vertex of each clique uncolored. Wang's Lemma 3.5 and Corollary 3.6 establish this principle more generally and count the choices within each group by its size. We apply that principle to this explicit graph and classify the remaining possibilities by the numbers of uncolored vertices on each side of the join. Two independent exact programs confirm the resulting polynomial.

This work has not yet been reviewed by experts. The proofs, the certificate checkers, the figure and the text were produced by AI models (from OpenAI and Anthropic), working under the author's direction.

MSC 2020: 05C31, 05C57. **Keywords:** zero forcing, graph polynomial, unimodality.

1 Introduction

For a finite simple graph H, color an initial set S blue and all other vertices white. A blue vertex with exactly one white neighbor may color that neighbor blue. The set S is a *zero-forcing set* if this rule can color every vertex. Write $z(H;k)$ for the number of such sets of size k, and

$$Z(H;x) = \sum_{k=0}^{|V(H)|} z(H;k)x^k.$$

A sequence is unimodal if it is weakly increasing up to some position and weakly decreasing thereafter. Let $\sqcup$ denote disjoint union and $\vee$ denote graph join: the join adds every edge between its two operands.

Theorem 1. *The graph*

$$G = (K_{32} \sqcup K_{33}) \vee (K_2 \sqcup K_2 \sqcup K_2 \sqcup K_2)$$

has 73 vertices and zero-forcing polynomial

$$Z(G;x) = x^{69}(2096 + 1592x + 1600x^2 + 73x^3 + x^4).$$

In particular, the zero-forcing polynomial of a finite simple graph need not be unimodal.

Boyer, Brimkov, English, Ferrero, Keller, Kirsch, Phillips and Reinhart asked whether the zero-forcing polynomial of every graph is unimodal [1, Question 1]. The 2019 article states this as a question; the 2018 preprint states the affirmative assertion as Section 6, Conjecture 1. Their Theorem 16 in that preprint proves $z(H;i) \le z(H;i+1)$ for $1 \le i < |V(H)|/2$. Thus the coefficients do not decrease through the first half; the question is whether they can decrease and then increase again later. The decrease and subsequent increase in Theorem 1 occur beyond that range.

The six displayed cliques are modules: vertices in one clique have identical neighbors outside it. Wang's Lemma 3.5 and Corollary 3.6 in [2] imply that a zero-forcing set leaves at most one vertex of each such clique uncolored. One can therefore count by choosing the cliques containing an uncolored vertex and weighting each choice by the product of their sizes. Wang treats the more general leaky forcing rule; ordinary zero forcing is its case with no leaks. Section 7(3) of that work asks about unimodality for graph classes with a bounded number of modules of identical outside neighbors. Our graph has six such modules. We add the explicit graph and its polynomial, whose coefficients fall and then rise: $2096 > 1592 < 1600$. The elementary specialization is included below, and Section 3 describes the accompanying computer checks.

The proof first excludes two uncolored vertices in one module, then counts by how many remain on each side of the join. Choosing one on the left contributes a factor $32 + 33 = 65$; choosing one in each left module contributes $32 \cdot 33 = 1056$. The latter contribution occurs only at degree 71, giving the increase from 1592 to 1600. We make no claim that 73 is the least order of a graph with this property.

Related work. Makowsky and Rakita [4, Conjecture 1] conjecture that, for every upward monotone augmented graph property, the sequence counting the distinguished sets of each size is unimodal for almost all graphs; their Examples 1.10(ii) place the zero-forcing polynomial in this setting. A single example does not bear on that question. Jacob [5] counts, for a graph H on n vertices, the sizes i with $0 < z(H;i) < \binom{n}{i}$, the zero forcing span, and lists graphs for which there is exactly one such size (Proposition 13 of the preprint); unimodality is not discussed. Menon and Singh [11] and German [12] study a different question of [1], stated as [11, Conjecture 1.1]: $z(H;i) \le z(P_n;i)$ for every graph H on n vertices and all $i \ge 1$, where P_n is the path. German mentions unimodality only as a question of [1]. We read both preprints in full; the published abstracts agree with them, up to two words in the first, and the reference lists are the same. Four further papers were read only in part, and each concerns a minimum size rather than the number of forcing sets of each size. Brimkov, Mikesell and Hicks [6] compute the zero-forcing number with satisfiability models, heuristics and polyhedral methods, as their abstract and the survey [7] describe. Karbasioun and Ameri's abstract [8] describes bounds and an algorithm for the zero-forcing number of fuzzy graphs. Karst, Shen, Troxell and Vu [9] and Beaudouin-Lafon and coauthors [10] bound the zero blocking number, the minimum size of a fort, of Cartesian products and of grids; we read their abstracts, introductions, section headings and references. Beaton's seminar abstract [13] announces conditions under which polynomials generated by clutters are unimodal; we found no slides or paper. Cameron's list of open problems on graph polynomials [14], last modified October 15, 2026, still lists unimodality of the zero-forcing polynomial as open. To our knowledge the question was open, and the result is new as far as we could check.

2 Counting the forcing sets

Call the cliques of orders 32 and 33 the *left modules*, and the four edges the *right modules*. Figure 1 shows their adjacencies. We count a zero-forcing set S through its complement $W = V(G) \setminus S$, the

initially white set. A *fort* is a nonempty vertex set F such that no vertex outside F has exactly one neighbor in F. Fast and Hicks introduced forts [3]; every zero-forcing set meets every fort, as also stated in [2, Proposition 2.3].

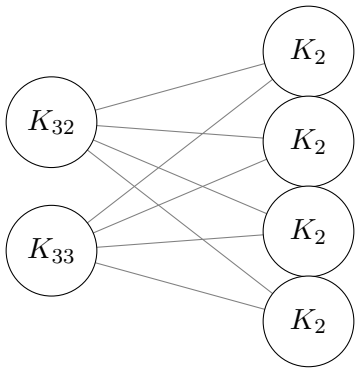


Figure 1: Each circle is a clique; each line denotes all edges between its endpoint cliques. No other intermodule edges occur.

Lemma 2 (Wang [2, Lemma 3.5], specialized)**.** *The complement of a zero-forcing set of G contains at most one vertex from each module.*

This is Wang's result with no leaks, applied to the six cliques; the proof is included for completeness. The same twin-pair observation appears in German's preprint [12, Section 3, Lemma 7].

Proof. Any pair of vertices in a module is a fort: each vertex outside the pair is adjacent either to both members or to neither. If a fort is initially white, consider a putative first force into it. Its blue source lies outside the fort and has a neighbor in it, hence at least two, all still white. This contradicts the color-change rule. Thus a forcing set cannot leave two vertices white in the same module. □

Every module has at least two vertices. Once the white set has at most one vertex per module, each module therefore contains a blue vertex. Let r be the number of white vertices in the left modules and s the number in the right modules. Thus $0 \le r \le 2$ and $0 \le s \le 4$.

Lemma 3. *An initial set with at most one white vertex per module is forcing exactly in the following cases:*

$$r = 0; \qquad r = 1 \text{ and } s < 4; \qquad r = 2 \text{ and } s = 0.$$

Proof. If $r = 0$, the blue partner of each white vertex in a right module has that vertex as its unique white neighbor. These forces finish the process, including the vacuous case $s = 0$.

If $r = 1$ and $s < 4$, a completely blue right module exists. Any vertex in that module has the unique white left vertex as its only white neighbor and forces it. The preceding case then applies. If instead $r = 1$ and $s = 4$, the blue vertex in each right module has two white neighbors, one left and one in its own module. A blue vertex in the left module containing the white left vertex has five white neighbors: that vertex and the four white right vertices. A blue vertex in the other left module has four white neighbors. There is no force, and the initial set is not forcing.

If $r = 2$ and $s = 0$, a blue vertex in either left module has its own module's white vertex as its unique white neighbor; the left modules are not adjacent. Both white vertices can be forced.

Finally, if $r = 2$ and $s \geq 1$, every blue vertex in a left module has $1 + s \geq 2$ white neighbors, and every blue right vertex has at least the two white left neighbors. Again no force is possible. These cases exhaust the ranges of r and s. □

Proof of Theorem 1. For a prescribed set of modules containing white vertices, the number of white sets is the product of their sizes. Lemma 2 excludes every other white set. The three cases of Lemma 3 are disjoint. If $r = 0$, choosing s right modules and one endpoint in each gives $\binom{4}{s}2^s$ white sets. If $r = 1$, the left choice has $32 + 33 = 65$ possibilities, and s ranges only through 3. If $r = 2, s = 0$, there are $32 \cdot 33 = 1056$ white sets. Hence

$$\begin{aligned} Z(G;x) &= \sum_{s=0}^{4} \binom{4}{s} 2^s x^{73-s} + 65 \sum_{s=0}^{3} \binom{4}{s} 2^s x^{72-s} + 1056x^{71} \\ &= x^{69}(2096 + 1592x + 1600x^2 + 73x^3 + x^4). \end{aligned}$$

For example, the three coefficients at the valley are $16 + 65 \cdot 32 = 2096$, $32 + 65 \cdot 24 = 1592$, and $24 + 65 \cdot 8 + 1056 = 1600$. All coefficients below degree 69 vanish. Since $2096 > 1592 < 1600$, no position can be the peak of a unimodal coefficient sequence. □

3 Verification and reproducibility

The file `data/certificate.json` lists the vertices and edges of G, and two independent exact programs compute $Z(G;x)$ from it. One checks ordinary forcing on each of the 64 choices of which modules contain a white vertex, weighted by module sizes. The other checks forcing on all 90,882 white sets with at most one vertex per module, counting each successful set directly. Both give the displayed polynomial, totaling 5,362 zero-forcing sets. A third check confirms that every other white set contains a within-module pair that is a fort. Separate checks reproduce the known polynomials of K_6, P_4 and C_4 and reject altered inputs. The command `python3 -I checks/check.py` runs these checks in seconds with standard Python; `checks/README.md` explains the inputs and the separate checks.

Code and data availability. The certificate `data/certificate.json`, the checks in `checks/` and this paper's source are available at

https://github.com/6/clank-lab/tree/zero-forcing-unimodality-v1/results/zero-forcing-unimodality

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