The weak parking-space conjecture

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Abstract

Armstrong, Reiner and Rhoades conjectured in 2012 that the noncrossing and algebraic parking spaces of an irreducible finite real reflection group have the same character under the reflection group and Coxeter rotation. We prove their weak conjecture in types E_7 and E_8, its last two open real types, completing the conjecture. The proof expresses fixed parking functions as intersections of conjugacy classes with cosets of pointwise parabolic subgroups. Complete finite certificates verify these intersections and the noncrossing interval, using exact arithmetic. Together with the previously established types, the result gives the conjectured complex representation isomorphism for every irreducible finite real reflection group. The assertion concerns the original parking spaces, with Fuss parameter 1.

1 Introduction

We complete the weak form of the parking-space conjecture of Drew Armstrong, Victor Reiner and Brendon Rhoades, first posted on April 8, 2012 [1]. Let W be an irreducible finite real reflection group on a real vector space V, let c be a Coxeter element, let h be its order, and put $C = \langle c \rangle$ and $\omega = \exp(2\pi i/h)$. A noncrossing flat is the fixed space of an element of the absolute-order interval $[1, c]$; this interval is defined in Section 3. For a flat X, let P_X be its pointwise stabilizer in W. The noncrossing parking set is

$$\mathrm{Park}_W^{\mathrm{NC}} = \{[x, X] : X \text{ is a noncrossing flat, } xP_X \in W/P_X\},$$

where $[x, X]$ records the coset xP_X as well as X. Its action is $(w, c^d)[x, X] = [wxc^{-d}, c^d X]$.

Theorem 1.1. *For W of type E_7 or E_8, every $w \in W$ and every integer d satisfy*

$$|\mathrm{Fix}_{\mathrm{Park}_W^{\mathrm{NC}}}(w, c^d)| = (h+1)^{\dim_{\mathbb{C}} \ker(w - \omega^d I | V \otimes_{\mathbb{R}} \mathbb{C})}. \tag{1}$$

Here $h = 18$ in type E_7 and $h = 30$ in type E_8.

The contribution is the two exceptional-type character computations requested in Armstrong–Reiner–Rhoades [1, Problem 11.1]. Their character formula and reduction to these remaining types are reused. The main computational difficulty is counting fixed cosets with Coxeter rotation still present: counting only the W-action omits the condition for nontrivial c^d. The reduction below retains every cyclic exponent, and compresses the count using cyclic orbits of flats and conjugacy

The proofs, the computation programs, the certificate checkers and the text were produced by AI models (from OpenAI and Anthropic), working under the author's direction.

classes of their parabolic normalizing subgroups. A separate exhaustive downward-closure check establishes that no noncrossing flat is omitted.

To identify the exact conjecture, let $\mathrm{Park}_W^{\mathrm{alg}}$ denote the algebraic parking space of [1, Section 2.5], described in Section 3. With the hypotheses on W and C above, the concluding sentence of their weak Main Conjecture is:

> "Then one has an isomorphism of $W \times C$-representations"
>
> $$\mathbb{C}[\mathrm{Park}_W^{\mathrm{alg}}] \cong_{\mathbb{C}[W\times C]} \mathbb{C}[\mathrm{Park}_W^{\mathrm{NC}}].$$

This is equation (2.7) of [1, Section 2.6]; the statement already appears in the first public version. The bracket notation in this quotation is retained from that source.

Corollary 1.2. *The weak Main Conjecture of Armstrong, Reiner and Rhoades holds for every irreducible finite real reflection group: its algebraic and noncrossing parking spaces are isomorphic as complex $W \times C$-representations.*

Table 1 gives the published results used for the other types, with the precise locations in [1]. Christian Stump's software established the earlier exceptional-type computations, as credited immediately after that paper's Table 1. Brendon Rhoades subsequently proved the intermediate conjecture in type A for every Fuss parameter and related it to the generic strong form [7, Theorems 1.1–1.2 and Table 1]. His earlier *Parking structures: Fuss analogs* [6] developed the Fuss generalization. Those stronger and more general results are not needed for the original parameter considered here.

Table 1: The prior-type bridge for Corollary 1.2. Strong and intermediate forms each imply the weak form. All references in the last column are to [1].

Types	Previously established form	Location
A_1, $I_2(m)$	Strong	Corollary 2.14
A_r	Weak	Section 8, equation (8.1)
B_r/C_r	Intermediate	Section 6.2
D_r	Intermediate	Section 7.2
H_3, H_4, F_4, E_6	Weak, by computation	Table 1 and following paragraph

Several nearby results distinguish why the cyclic factor matters. Theo Douvropoulos and Matthieu Josuat-Vergès established the noncrossing parking character for the W-action in all real types, including E_7 and E_8, for arbitrary Fuss parameter [4, Section 3.2, Theorem 3.10]; the discussion in that section explicitly separates the cyclic enhancement. Yosuke Ito and Soichi Okada classify when generalized parking class functions are W-characters or permutation characters [5, Theorem 1.4 and Section 7.2]. These existence statements do not identify the prescribed noncrossing $W \times C$-action. Jason Stack extends the parking-space construction to well-generated complex reflection groups, with G_{34}, E_7 and E_8 expressly excluded from the hypotheses of his main results [8, Theorems 1.3 and 3.19]. Jason Stack and Nathan Williams study homogeneous systems of parameters and the resulting graded and ungraded W-modules [9, Section 1.2]; their credit to Douvropoulos–Josuat-Vergès for the E_7, E_8 noncrossing character concerns the W-action. Theorem 1.1 supplies the full cyclic character comparison in these two real types.

The conclusion is equality of complex characters and hence a representation isomorphism. It does not supply an equivariant bijection of the underlying sets, prove the intermediate or strong

conjecture, treat arbitrary Fuss parameter, or address G_{34}. The proof is computer-assisted and has not yet been reviewed by experts.

Section 2 gives the proof strategy. Section 3 fixes the reflection and parking-space definitions. Section 4 derives the finite counting formula and proves the interval-cover criterion. Section 5 describes the certificates, the exact comparisons and the trusted computational boundary.

2 Proof overview

The proof reduces the large parking sets to small group-theoretic counts. The algebraic side is already known: Armstrong–Reiner–Rhoades give its value as the right side of (1) [1, Proposition 2.15]. It therefore suffices to compute every value of the noncrossing parking character.

Fix a noncrossing flat X. A parking point over X can be fixed by (w, c^d) only if $c^d X = X$. In that case its coset condition becomes $x^{-1}wx \in P_X c^d$. Consequently the contribution over X is a centralizer factor times the size of the intersection of a W-conjugacy class with $P_X c^d$. Flats in the same C-orbit make equal contributions. If this orbit has length ℓ, the group $K = \langle P_X, c^\ell \rangle$ contains P_X as a normal subgroup with cyclic quotient. The coset $P_X c^d$, when relevant, is a union of K-conjugacy classes. Thus complete class lists for W and these groups K suffice to compute the entire character.

There are two completeness obligations. First, the list of noncrossing elements must be the whole absolute-order interval, not just a collection with the expected cardinality. Starting from c, closure under every reflection step that lowers reflection length proves this by a reduced-factorization argument. Second, the supplied class representatives must cover W and each K. Centralizer-derived class sizes, pairwise nonconjugacy, and their sum establish these covers. Conjugating witnesses then determine how the K-classes lie in W-classes, without relying on class labels.

Two implementations check these certificates and recompute all character values. Both use the same exact GAP group kernel. The mathematical reduction and that common computational trust are stated explicitly below.

3 Reflection and parking spaces

This section relates the finite objects counted in the proof to the parking spaces in the conjecture.

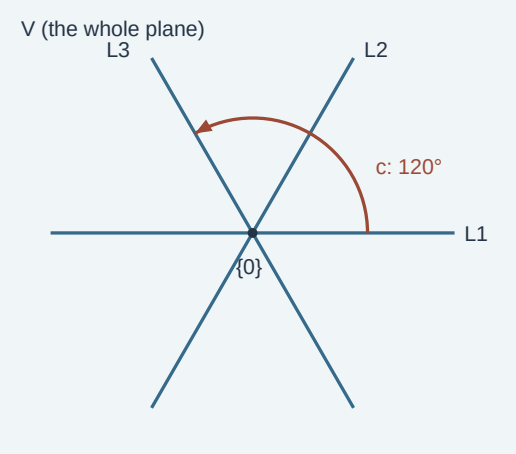
As a small example, in type A_1 the group is $W = \{1, s\}$ acting on a line, and $c = s$. There are two noncrossing flats, the whole line V and $\{0\}$. Their pointwise stabilizers are $\{1\}$ and W, respectively, so the parking set has two points over V and one over $\{0\}$. The element $(1, s)$ fixes only the point over $\{0\}$; the element (s, s) fixes all three. These are the values 3^0 and 3^1 in (1).

In type A_2, the group is the symmetry group of an equilateral triangle. Its five noncrossing flats are the plane, the three reflection lines, and the origin. Figure 1 shows these flats, their Coxeter rotation, and the sizes of their coset fibers. The parking points are the cosets over each flat, rather than points drawn on a line.

Let T be the set of reflections of W. The reflection length $\ell_T(g)$ is the least number of factors in an expression of g as a product of elements of T. Absolute order is defined by

$$u \le_T v \quad \Longleftrightarrow \quad \ell_T(v) = \ell_T(u) + \ell_T(u^{-1}v), \qquad \mathrm{NC}(W, c) = [1, c]_T.$$

Carter's formula gives $\ell_T(g) = \mathrm{rank}(I - g)$ in the real reflection representation [3, Section 2]; see also [1, Section 2.1]. The map $u \mapsto \mathrm{Fix}_V(u)$ is injective on $\mathrm{NC}(W, c)$ by the noncrossing-flat embedding of Brady and Watt [2, Theorem 2], used in [1, Definition 2.2]. Its image is the set of noncrossing



Parking fibers over the five flats

The fiber over X is the coset set W/P_X.

Flat X	|P_X|	|W/P_X|
V	1	6
L1, L2, L3	2	3 each
{0}	6	1

6 + 3 + 3 + 3 + 1 = 16 parking points

The drawing shows flats; the counts refer to cosets.

Figure 1: The type-A_2 parking construction. Coxeter rotation permutes the three lines. The fiber over a flat X is W/P_X; its size is shown on the right. This small example illustrates the definition; the exceptional-type proof uses the separate certificates below.

flats. With P_X the pointwise stabilizer, the parking set and action in the introduction agree with [1, Definition 2.6 and Section 2.3].

Write $n = \dim_{\mathbb{R}} V$. For the algebraic space, choose coordinates $x_1, \dots, x_n$ on the complexified reflection representation and a homogeneous system of parameters $\Theta = (\theta_1, \dots, \theta_n)$ of degree $h+1$ for which the linear map $x_i \mapsto \theta_i$ from the dual reflection representation is W-equivariant. Here a homogeneous system of parameters means homogeneous polynomials for which the quotient by their ideal is finite-dimensional. The algebraic parking module is

$$\mathbb{C}[V \otimes_{\mathbb{R}} \mathbb{C}]/(\theta_1 - x_1, \dots, \theta_n - x_n),$$

with the W-action on polynomials induced by its action on V, and with c acting on a homogeneous polynomial of degree j by ω^{-j}; see [1, Section 2.5, equation (2.4) and Definition 2.10]. Their Propositions 2.11 and 2.15 show that its character is independent of the permitted choice of Θ and equals $(h+1)^{\dim_{\mathbb{C}} \ker(w - \omega^d I)}$ at (w, c^d). This is the only algebraic input needed in the comparison.

The finite model in type E_n, for $n = 7, 8$, uses the simple-root Gram matrix A with diagonal entries 2 and off-diagonal entry -1 on the edges

$$(1,3),\ (3,4),\ (2,4),\ (4,5), \dots, (n-1, n),$$

and zero elsewhere. Let e_i denote the ith standard coordinate row vector. In simple-root coordinates the simple reflections on row vectors are

$$s_i(v) = v - (vAe_i^{\mathsf{T}})e_i.$$

Their orbits on the simple roots give the root system; permutation of all roots is faithful because the simple roots span the representation. The element used for the calculation is $c = s_1 \cdots s_n$. All Coxeter elements in a finite real reflection group are conjugate, so transport by conjugation gives the assertion for any choice of c.

4 The finite counting reduction

We now derive the character formula used by the checkers and establish the finite cover that makes its computation exhaustive.

Write $\mathrm{Cl}_W(w)$ for the conjugacy class of w and $C_W(w)$ for its centralizer. An orbit of flats means an orbit under C.

Proposition 4.1. *Choose one representative X from each orbit of noncrossing flats, write ℓ_X for its orbit length, and put $K_X = \langle P_X, c^{\ell_X} \rangle$. For every $w \in W$ and integer d,*

$$\left|\mathrm{Fix}_{\mathrm{Park}_W^{\mathrm{NC}}}(w, c^d)\right| = \sum_{\substack{X \text{ an orbit representative} \\ \ell_X \mid d}} \frac{\ell_X \, |C_W(w)|}{|P_X|} \left|\mathrm{Cl}_W(w) \cap P_X c^d\right|. \tag{2}$$

Every intersection in the sum is a disjoint union of K_X-classes.

Proof. A fixed parking point must lie over a flat fixed by c^d. When $c^d X = X$, equality of the cosets of $[wxc^{-d}, X]$ and $[x, X]$ is equivalent to $x^{-1}wx \in P_X c^d$. The map $x \mapsto x^{-1}wx$ from W to $\mathrm{Cl}_W(w)$ has constant fiber size $|C_W(w)|$. Dividing by the $|P_X|$ representatives of each coset gives

$$\frac{|C_W(w)|}{|P_X|} \left|\mathrm{Cl}_W(w) \cap P_X c^d\right|$$

fixed parking points over X.

The flat is fixed by c^d exactly when ℓ_X divides d. Conjugation by powers of c carries P_X to the stabilizers of the other flats in its orbit, preserves $\mathrm{Cl}_W(w)$, and preserves c^d. Thus all ℓ_X flats make the same contribution. This proves (2).

The element c^{ℓ_X} normalizes P_X, since it preserves X. Hence P_X is normal in K_X and K_X/P_X is cyclic. If $\ell_X \mid d$, then $c^d \in K_X$. Every coset of P_X in K_X is invariant under conjugation in K_X, because that quotient is abelian. The intersection of such a coset with a W-class is therefore a union of whole K_X-classes. □

The pointwise stabilizer P_X can be constructed without recognizing its parabolic type: it is generated by all root reflections whose hyperplanes contain X. This is the real parabolic stabilizer theorem. For completeness, choose a real point of X outside the finitely many proper subspaces $X \cap \mathrm{Fix}(g)$ with $g \notin P_X$. Its stabilizer is exactly P_X. The chambers incident to this point are joined by crossings of walls through the point. Their wall reflections generate its stabilizer, since W acts simply transitively on chambers. A reflection fixes the chosen point precisely when its hyperplane contains X. The construction of P_X therefore uses all and only the required reflections.

Lemma 4.2 (Complete interval cover)**.** *Let S be a subset of $[1, c]_T$ containing c. Suppose that, for every $g \in S$ and reflection $t \in T$ with $\ell_T(gt) = \ell_T(g) - 1$, the element gt belongs to S. Then $S = [1, c]_T$.*

Proof. For $u \leq_T c$, concatenate reduced reflection factorizations of u and $u^{-1}c$. Their concatenation is a reduced factorization of c. Deleting its rightmost factors until the prefix for u remains gives a descending path from c to u, each step a right reflection multiplication lowering length by one. Every prefix lies in $[1, c]_T$. The closure assumption forces this entire path, including u, into S. □

By Carter's formula, all lengths in this lemma and interval membership are checked by exact ranks. For example, membership is the equality

$$\mathrm{rank}(I - g) + \mathrm{rank}(I - g^{-1}c) = n.$$

The expected generalized Catalan cardinality is an additional crosscheck; it does not replace Lemma 4.2.

Action conventions

This paragraph makes explicit the identification between the row-matrix calculation and the left geometric action in the statement. GAP permutations act on the right, and its conjugation notation is $g^a = a^{-1}ga$. If $M(g)$ is the computed row matrix, the associated left action is $v \mapsto vM(g)^{-1}$. Its fixed spaces equal those of $M(g)$. The left Coxeter action traverses the flat orbits in the opposite direction from the row-matrix traversal; their members and lengths are unchanged. The abstract left-coset condition remains $x^{-1}wx \in P_X c^d$. Finally, the matrices are real, so the multiplicities of any eigenvalue and its complex conjugate agree. Inverting $M(w)$ therefore leaves the multiplicity of ω^d unchanged. These facts account for the direction and inverse conventions in both sides of (1).

5 Certificates and exact computation

This section states the finite computational results used in the proof, explains how a reader checks them, and specifies what is trusted.

5.1 What the certificates contain

Each type has an uncompressed JSON file containing the group and character data, and a compressed JSON certificate containing an interval enumeration, the cyclic orbits, and the subgroup-class data. The files are in `data/` in the accompanying code. Compression is only lossless storage.

The interval is encoded by parent edges. Its first element is c; each later element is obtained from an earlier one by a specified root reflection. A checker reconstructs these elements, checks their distinctness and interval membership, and checks every descending reflection step in Lemma 4.2. It also computes the fixed spaces by rational row reduction, checks that they are distinct, and checks that the specified cyclic orbits partition them.

For W and each K_X, the class lists are checked by group membership, exact centralizer-derived class sizes, and pairwise nonconjugacy. Their sizes must sum to the order of the group. These conditions prove that the lists cover the entire group. Every supplied subgroup-class representative has a conjugating witness in W that identifies its W-class. The checkers verify this witness; they do not infer fusion from a class name, its order, or its eigenvalues.

For each exponent d, membership in $P_X c^d$ determines which K_X-classes contribute. Their sizes, with the factors in Proposition 4.1, give the parking character. Independently, exact nullspaces over cyclotomic fields give the eigenvalue multiplicities. The checkers compare the two integers for each pair consisting of a W-class and an exponent $0 \le d < h$.

Proposition 5.1 (Finite computation)**.** *For the standard root representations of E_7 and E_8, the certificates pass both full checkers. In every W-conjugacy class and for every exponent $0 \le d < h$, formula (2) equals the right side of (1). There are no unresolved entries.*

Table 2 records the size of these finite checks. The E_6 data are included as a previously solved control; they are not a new case. The subgroup-class row is the sum over the groups K_X attached to the orbit representatives.

Proof of Theorem 1.1. Lemma 4.2 and the checked class covers make the calculation of Proposition 4.1 exhaustive. Proposition 5.1 establishes (1) on every W-class for $0 \le d < h$. Both sides are constant on W-classes and periodic in d modulo h, so this proves all the asserted equalities. The action-convention identification above gives the specified reflection representation and parking action. □

Table 2: Exact certificate checks.

Quantity	E_6 (control)	E_7	E_8
Coxeter number	12	18	30
Group order	51 840	2 903 040	696 729 600
Roots	72	126	240
Noncrossing elements	833	4 160	25 080
Cyclic orbits of flats	83	468	1 680
W-conjugacy classes	25	60	112
Subgroup-class rows	691	8 428	40 172
Character comparisons	300	1 080	3 360

Proof of Corollary 1.2. Theorem 1.1 supplies the two exceptional types absent from Table 1. The classification of irreducible finite real reflection groups makes this list exhaustive. Proposition 2.15 of [1] identifies the right side of (1) with the algebraic parking character. Equality of complex characters of a finite group is equivalent to isomorphism of the corresponding complex representations. □

5.2 Reproducing the check

The accompanying README gives installation instructions. From the result's directory, with GAP and Python and the packages in `checks/requirements.txt` installed, the complete command is

```
python3 checks/verify.py
```

This checks the committed certificates for E_6, E_7, E_8 with both implementations; it does not regenerate them with the producer. It also checks exact eigenvalue data. As an additional crosscheck, it compares the cyclic fixed-flat counts with root-of-unity evaluations of the q-Catalan polynomial described in the checker documentation. The computation uses no floating-point tests and requires no CHEVIE package or character-table labels. The producer is included so that the certificates can also be regenerated, separately from their verification.

Two deliberate corruptions test rejection. A changed character value is rejected by both checking paths; the second path detects it in its exact eigenvalue check. A genuine interval gap is formed by removing the final identity leaf in the E_6 parent list and decreasing the recorded count, so mere length consistency cannot detect it. Both GAP implementations reject this missing descending neighbor. These controls test the stated failure modes; the exhaustive positive check is the evidence for the theorem.

5.3 Trusted boundary

The proof uses ordinary reflection-group mathematics: the standard E_n root representation, Carter's reflection-length formula, Brady–Watt's noncrossing-flat embedding, the real parabolic stabilizer theorem, the previously established types in Table 1, the algebraic character formula of Armstrong–Reiner–Rhoades, and finite-group character theory. The counting and cover arguments in Section 4 are written mathematical arguments.

The full checkers trust GAP's exact integer, rational and cyclotomic arithmetic; its permutation multiplication, membership and group-order algorithms; its centralizers and conjugacy tests; and its exact rank, nullspace and row-reduction routines. Their certificate logic is separate, but both

full checkers share the GAP kernel. They are not two independent implementations of that kernel. The Python transport and JSON/gzip decoding are also trusted, as is SymPy's exact rational polynomial arithmetic in the additional spectral checks. Production class enumeration and conjugating witnesses are checked by the class covers and witness tests described above. No claim of a complete Lean proof or a kernel-checked certificate-to-theorem bridge is made.

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