The convex uniform 5-polytopes: a computer-assisted classification

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Abstract

We prove completeness of the convex uniform 5-polytope list, conditional on the cited lower-dimensional classifications, the written reductions and the certificate checkers. Its members are Wythoff's constructions of types A_5, B_5, D_5, prisms over uniform 4-polytopes, and products of regular polygons with uniform polyhedra. Conway and Guy announced the four-dimensional classification; Möller gave a complete computer-assisted proof. We extend these neighborhood and radius methods to dimension five. Coxeter's and Johnson's constructions supply the known families. Unbounded polygon sizes require products, forced by neighbor distances and prism cells. Lean checks a conditional implication from explicit premises.

1 Introduction

All polytopes are convex. Uniform polygons are regular; in higher dimensions uniformity means vertex transitivity and uniform facets. All edges have unit length. Write $\{p\}$ for a regular polygon, $\{\} = [0,1]$, $P_q = \{q\} \times \{\}$, and A_q for an antiprism. Coxeter and antiprism subscripts mean rank and polygon size, respectively.

Main Theorem. *Assume Facts 2 and 3. Every convex uniform 5-polytope is congruent, after scaling to unit edges, to one of the following:*

(K1) *a Wythoffian polytope W_J of one of the Coxeter groups A_5, B_5, D_5 (19 distinct A_5 forms, 31 B_5 forms and 8 further D_5 forms);*

(K2) *a prism $K \times \{\}$ over a convex uniform 4-polytope K; 45 Wythoffian bases, plus the snub 24-cell and grand antiprism, give 47 prisms, of which 46 are not already in (K1), since the tesseract prism is the 5-cube;*

(K3) *a product $\{p\} \times Q$ of a regular polygon, $p \geq 3$, with a convex uniform polyhedron Q: a Platonic or Archimedean solid, a prism P_q (giving the* duoprism prisms $\{p\} \times \{q\} \times \{\}$*), or an antiprism A_q.*

The proof below uses written reductions and certified computations; the formal theorem takes their conclusions as parameters.

Definition 1 (Wythoffian polytopes [1, 2])**.** Let W be a finite irreducible Coxeter group of rank k acting on $\mathbb{R}^k$, with simple roots $\alpha_0, \ldots, \alpha_{k-1}$ (nodes of the Coxeter diagram), and let $\emptyset \neq J \subseteq \{0, \ldots, k-1\}$ (the *ringed* nodes). Let x_J be the point at distance $1/2$ from the mirrors of the ringed

nodes and on the mirrors of the others, on the positive side of all of them. The *Wythoffian polytope* $W_J = \mathrm{conv}(Wx_J)$ is uniform with unit edges. We name it by the group and the ring pattern, e.g. B_5 `00001`; node orders are those of Table 2 in Appendix B.

Fact 2 (3-D classification [21, Section 18.5], [13])**.** *A convex uniform polyhedron is a Platonic solid, one of the 13 Archimedean solids, a prism P_q ($q \geq 3$) or an antiprism A_q ($q \geq 4$). It is determined up to congruence by its vertex configuration (the cyclic sequence of polygon sizes around a vertex). In particular a convex uniform polyhedron with a q-gonal face, $q \notin \{3, 4, 5, 6, 8, 10\}$, is P_q or A_q.*

Fact 3 (4-D classification [8, 17])**.** *A convex uniform 4-polytope is one of: the Wythoffians of A_4, B_4, D_4, F_4, H_4; the snub 24-cell; the grand antiprism; a prism $Q \times \{\}$ over a convex uniform polyhedron Q; a duoprism $\{p\} \times \{q\}$. In particular a convex uniform 4-polytope with a q-gonal 2-face, $q \notin \{3, 4, 5, 6, 8, 10\}$, is a duoprism $\{q\} \times \{p\}$ (including $P_q \times \{\} = \{q\} \times \{4\}$) or the antiprism prism $A_q \times \{\}$.*

The configuration and large-polygon clauses follow by inspecting these lists.

Möller's complete four-dimensional proof [17, Section 1.1] and its published account [18] follow Conway–Guy's announcement [8, pp. 38–39]. Möller's radius calculations, edge neighborhoods and vertex completion [17, Chapters 4–7 and 9] are foundational. Finite neighborhood enumeration cannot bound polygon sizes. Our additional argument forces products for all large sizes. Schulte reports a complete list only for $d = 4$ [21, Section 18.5]; Winter records as unknown whether "there exist non-prismatic non-Wythoffian uniform polytopes in dimension five or above" [24, Appendix E.3, p. 167].

The Open Problem Garden also lists the five-dimensional enumeration question [19].

Proof overview. Facet cycles fix radii; completing incident facets determines the bounded cases. For large polygons, distance bounds leave one polygon or two sharing a duoprism at each vertex. Their prism cells propagate translates and force products.

The proofs, the Lean formalization and the text were produced by AI models (from OpenAI and Anthropic), working under the author's direction. This work has not yet been reviewed by experts.

2 Local geometry

Berestovskii–Nikonorov proved separation [25, Theorem 3.1]. Let P be five-dimensional, with centroid c and circumradius R. Face centers are the orthogonal projections of c (Lemma 14).

Lemma 4 (Separation lemma)**.** *Let X be a convex polytope inscribed in a sphere, with at least three vertices, let v be a vertex and w a vertex nearest to v among the other vertices. Then $[v, w]$ is an edge of X. Consequently, in a convex uniform polytope with unit edges any two distinct vertices are at distance at least 1.*

Proof. The diameter ball of $[v, w]$ contains no other vertex. Subtracting its power function from the circumsphere's gives a supporting affine function vanishing exactly at v, w. □

For a p-gonal face G with center o_G, put $\rho = r_p = 1/(2\sin(\pi/p))$. In its plane's orthogonal complement N, incident cells (3-faces) give cyclic rays u_{C_i}; consecutive cells bound a facet with dihedral angle α_i.

Lemma 5 (Closure equation)**.** *With this notation, let* $w = c - o_G \in N$, $h = |w| = \sqrt{R^2 - \rho^2}$, *and for each cell* $C \supset G$ *of circumradius* R_C *let* $d_C = \sqrt{R_C^2 - \rho^2} > 0$. *Then:*

(a) $\sum_i \alpha_i < 2\pi$;

(b) $w \cdot u_C = d_C$, *so the angle* δ_C *between* w *and* u_C *satisfies* $\cos\delta_C = d_C/h$;

(c) w *lies strictly inside the cone spanned by the* u_C, *and with* $\theta_i \in (0,\pi)$ *the angle at* $\hat{w} = w/h$ *of the spherical triangle* $(\hat{w}, u_{C_{i-1}}, u_{C_i})$,

$$\cos\theta_i = \frac{\cos\alpha_i - \cos\delta_{C_{i-1}}\cos\delta_{C_i}}{\sin\delta_{C_{i-1}}\sin\delta_{C_i}}, \qquad \sum_{i=1}^{k}\theta_i = 2\pi;$$

(d) $R > R_K$ *for every facet* $K \supset G$.

Proof. The spherical section has perimeter below 2π. Projection gives (b). The interior direction partitions the full angle into spherical triangles; their cosine rule gives (c). Positive distance to the facet gives (d). □

The vertex figure VF is the neighbors' convex hull, of radius $\sqrt{1 - 1/(4R^2)}$. Its edges have lengths $\mathrm{ch}_p = 2\cos(\pi/p)$. A *star* consists of the full incident facets. Fix R, the center and a facet's base vertex and neighbors. These span and fix the full facet. Across a cell, convexity fixes the next hyperplane; its ordered correspondence and center fix the isometry. Traversal fixes the star (Lemma 26). If each neighbor's star is forced, transporting by vertex symmetries and inducting along the connected edge graph identifies P [5] (Lemma 19).

3 Bounded polygon sizes

Assume all polygons have at most 16 sides. Let $\mathcal{K}_{16}$ be (K1)–(K3) with polygon sizes at most 16.

Theorem 6 (Case A)**.** *Every convex uniform 5-polytope all of whose 2-faces have at most 16 sides is congruent to one of the 628 distinct polytopes of the list* $\mathcal{K}_{16}$ *of Section 3 (all members of (K1)–(K3) whose polygons have at most 16 sides).*

Proof. The classifications give a finite facet list. The closure equation enumerates cyclic facet words with angle sum at most 2π. Remove exact flat and degenerate cases; let Ω be the union of remaining radius intervals (Computation 38). Overlap components collect possible common radii. Repeatedly remove cycles missing a required facet incidence. A cycle of P remains: every polygon orbit of a facet occurs through its base vertex (Lemma 22).

Fix the least facet and its position. Every continuation across a ridge has the same cell label, some ordered correspondence and the opposite hinge. Exhaust these possibilities, identifying only exact isometries. Strict separation, support or incidence contradictions exclude cases; uncertain cases remain (Lemma 23). Computation 41 leaves 629 signatures, the multisets of facet types and cyclic edge words.

The closure function is the angle sum minus 2π, parametrized by reciprocal radius. For each signature it changes sign at the interval endpoints and has derivative excluding zero throughout, giving a unique radius (Computation 43). Every actual star has a represented ordered encoding. For each known construction, exact exclusions eliminate every other encoding; its existence supplies the star. Fixed-radius rigidity identifies it. Every continuation at a neighbor then gives that same star, hence that polytope.

The remaining signature has eight neighbors and $R = 1$. Its unique encoding forces the first neighbor's star; at the second, none of the 48 congruences contains all known facets. Thus it cannot occur (Computation 44). The enumeration gives 629 signatures: 628 known polytopes and this one refuted case. □

4 Large polygons

A large polygon has at least 17 sides. This exceeds the exceptional sizes in Facts 2–3 and satisfies $2\sin(2\pi/17) < 1$. At each vertex there are s large polygons, by transitivity. Each gives a long VF-edge e; its *fan* comprises the VF-facets containing it.

Lemma 7 (Long edges are disjoint)**.** *Two long VF-edges share no vertex.*

Proof. Endpoints opposite a shared endpoint would lie within $2\pi/17$ of its antipode, hence less than one apart. □

The angle sum allows three cycle types: (i) duoprisms $\{n\} \times \{q_i\}$ with three to five rays; (ii) two $A_n \times \{\}$ and one duoprism; (iii) two $A_n \times \{\}$ and two duoprisms, with pairs $(3,3), (3,4), (3,5)$. Indeed an antiprism ray costs two right angles; none or one occurs. At least one duoprism is necessary; two require $1/q_1 + 1/q_2 > 1/2$; three exceed the angle sum.

For type (i), lateral unit vectors satisfy $u_i \cdot (c - o_G) = 1/2$. Their circle radius $S \in (\max_i \cos(\pi/q_i), 1)$ uniquely solves

$$\sum_i 2\arcsin(\cos(\pi/q_i)/S) = 2\pi, \quad R_c^2 = \frac{1}{4(1-S^2)}, \quad R^2 = r_n^2 + R_c^2.$$

For (ii) and (iii), with antiprism height $h_A(n)^2 = 1 - 1/(4\cos^2(\pi/(2n)))$, projection gives respectively

$$R^2 = R_{A_n}^2 + r_q^2, \qquad R^2 = R_{A_n}^2 + \frac{1}{2(1+\gamma)}, \qquad R_{A_n}^2 = r_n^2 + h_A(n)^2/4.$$

Here γ is the cosine between the two lateral prism rays. The radius certificates separate these from Ω: the finite check ends at 397, and $r_{398} > \max \Omega$ covers every tail.

Corollary 8. *Every 2-face G of P lies in a large facet. Consequently every VF-edge at v lies in a VF-facet $\mathrm{VF}(K)$ with K large, and K contains a large 2-face through v.*

Proof. Otherwise its small-facet cycle puts R in Ω, contradicting the radius forced by a large polygon. Every vertex of a large facet lies on a large polygon. □

A fan is *paired* if a facet contains another large polygon. Configuration-radius comparisons exclude 67 unpaired configurations. The fan distance certificates bound pairs of admissible points, meaning points on the VF sphere satisfying the fan's supporting inequalities and separation from its vertices.

Lemma 9. *Let F be a large 2-face at v whose fan is in the list of Computation 48 (with the actual n, m of P). Then the only long VF-edges with an end among the fan vertices or an admissible point are e and, for a paired fan, the partner edge. In particular every long VF-edge is e or the partner edge.*

Proof. Disjointness excludes distinguished endpoints. The distance certificates leave only pairs of fan vertices. Other pairs are short or polygon diagonals. The lateral vertices lie on a circle in fan order; a diagonal crosses another diagonal internally, so cannot be a VF-edge. For three vertices there is no diagonal. □

Seven paired fans need $n = m$. Put $\delta_{c_1}(m) = R^2_{(m,c_1)} - r_m^2$. The partner cycle gives

$$R^2 = r_n^2 + r_m^2 + \delta_{c_1}(m), \quad \delta_{c_1}(m) = \delta_{c_2}(n) \text{ or } h_A(m)^2/4.$$

Finite comparison forces equality of parameters (Lemma 30), bringing these fans under the distance bound.

Proposition 10. *If $s \geq 2$, then $s = 2$ and the two large 2-faces F_1 (n-gon) and F_2 (m-gon) at v lie in a common facet $D = \{n\} \times \{m\}$.*

Proof. An unpaired fan permits only its distinguished long edge; a paired fan permits exactly it and its partner. Their common facet is the duoprism. □

Fact 11 (F3)**.** *In $\{n\} \times \{q\}$ and in $A_n \times \{\}$ every vertex lies on an n-gon. At a vertex v: $\mathrm{VF}(\{n\} \times \{q\})$ is the tetrahedron $\mathrm{conv}(a, b, g, g')$, where a, b are the neighbors on the n-gon and g, g' those on the q-gon: $|ab| = \mathrm{ch}_n$, $|gg'| = \mathrm{ch}_q$, $ab \perp gg'$, and the other four edges have length $\sqrt{2}$. $\mathrm{VF}(A_n \times \{\})$ is the pyramid with apex z (the prism edge) over the trapezoid $(a, b, y, x) = \mathrm{VF}(A_n)$: ab is the VF-edge of the n-gon, $|by| = |yx| = |xa| = 1$ (the three triangles at v), the diagonals ay, bx have length $\sqrt{1 + \mathrm{ch}_n}$, and the four apex edges have length $\sqrt{2}$.*

Proof. Product neighbors give these lengths, incidences and orthogonality. □

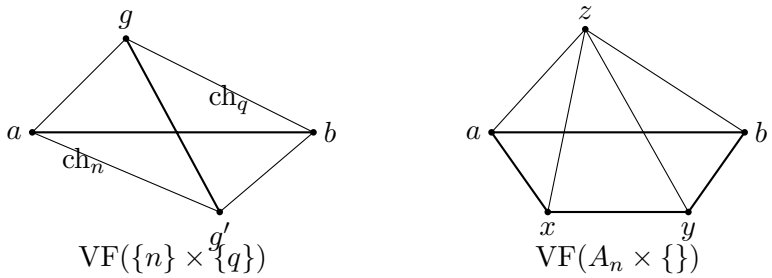


Figure 1: The two vertex figures of Fact 11, drawn schematically. Left: the tetrahedron, with orthogonal distinguished chords. Right: the pyramid over the antiprism's trapezoid; ab is the long edge.

Theorem 12. *If $s = 1$, then $P \cong \{n\} \times Q$ for a convex uniform polyhedron Q, or $P \cong A_n \times \{q\}$.*

Fix v, its large 2-face F (n-gon with plane direction L) and $e = [a, b]$ its VF-edge. By Corollary 8 and $s = 1$, every VF-edge lies in a VF-facet $\mathrm{VF}(K)$ with $K \supset F$; so *every VF-edge, and hence every VF-vertex, lies in a facet of the fan of e.* The fan is given by the cycle type of F (Lemma 27).

Type (i). The fan facets are $T_i = \mathrm{conv}(a, b, x_i, x_{i+1})$, where $[v, x_i]$ is the lateral edge of the i-th cell $F \times [v, x_i]$; so $x_i - v \perp L$, and the neighbors of v are exactly $a, b, x_1, \ldots, x_k$. Distinct prism cells give distinct lateral edges.

Product lemma. For a vertex t let F_t be its unique large 2-face. Transitivity gives the same local picture at t, carrying F to F_t. For a neighbor t' of t, either $t' \in F_t$, and then $F_{t'} = F_t$; or $t' - t$ is a lateral edge, orthogonal to the plane of F_t, and the duoprism $\{n\} \times \{q_i\}$ containing the edge tt' contains the translate $F_t + (t' - t)$, a large 2-face at t', so $F_{t'} = F_t + (t' - t)$. The edge graph of P is connected [5], so every F_t is a translate $F + \tau_t$ with $\tau_t \perp L$ (each step adds 0 or a vector orthogonal to L). Hence the vertex set is $F_0 + X$, with F_0 the polygon F centered at the origin of L and X a finite set in an affine copy of $L^\perp$, and $P = \operatorname{conv} F_0 + \operatorname{conv} X \cong \{n\} \times Q$ with $Q = \operatorname{conv} X$, a 3-polytope since $\dim P = 5$. A product point is a vertex exactly when both coordinates are vertices, so $X = \operatorname{Vert} Q$. Product faces give Q unit edges and regular faces. Symmetries preserve the unique polygon plane L, hence $L^\perp$, and induce a transitive isometry group on Q. Thus Q is uniform. □

Type (ii). The fan consists of the pyramids $\Pi_1 = (z_1; a, b, y, x)$, $\Pi_2 = (z_2; a, b, y, x)$ (the two $A_n \times \{\}$ sharing the A_n cell C, $\mathrm{VF}(C) = (a, b, y, x)$) and $T = \operatorname{conv}(a, b, z_1, z_2)$. The neighbors of v are a, b, x, y (in C) and z_1, z_2, with $z_j - v$ orthogonal to the 3-space M of C. *Cell partition:* for a vertex t let C_t be the unique A_n cell containing F_t (one A_n ray per cycle). For $t' \in C_t$, t' lies on exactly one of the two n-gons of C_t ($n \geq 4$), which is then $F_{t'}$, so $C_{t'} = C_t$. The other two neighbors t' of t satisfy $t' - t \perp M_t$, and the $A_n \times \{\}$ containing tt' gives $C_{t'} = C_t + (t' - t)$. So the vertex set is $C_0 + X$ with X in a 2-plane orthogonal to M, $P = A_n \times \operatorname{conv} X$, and $\operatorname{conv} X$ is a vertex-transitive equilateral convex polygon with unit edges, i.e. a regular polygon $\{q\}$. □

Type (iii) does not occur. The fan consists of Π_1, Π_2 as in (ii) and $T_1 = \operatorname{conv}(a, b, z_1, w)$, $T_2 = \operatorname{conv}(a, b, w, z_2)$, where $w - v$ is the lateral edge of the P_n cell between the two duoprisms. The seven points a, b, x, y, z_1, z_2, w are distinct: $a - v, b - v \in L$; $x - v$ and $y - v$ are neither in L nor orthogonal to L (x, y lie on the other n-gon of the antiprism, rotated by π/n and at height $h_A > 0$), and $x \neq y$; $w - v$, $z_1 - v$, $z_2 - v$ are orthogonal to L and are the lateral edges of three different P_n cells. Since every VF-edge lies in a fan facet, the VF-edges are exactly the edges of these four facets:

$$ab,\ ax,\ xy,\ yb,\ z_j a,\ z_j b,\ z_j x,\ z_j y\ (j = 1, 2),\ wa,\ wb,\ wz_1,\ wz_2.$$

In particular $z_1 z_2$ is not a VF-edge, and x has exactly the VF-edges xa, xy, xz_1, xz_2, y exactly yb, yx, yz_1, yz_2, and w exactly wa, wb, wz_1, wz_2; each of x, y, w is simple (Lemma 18). The triangle (x, y, z_1) is a 2-face of Π_1, hence a VF-ridge, lying in Π_1 and exactly one other VF-facet K'. At x, Π_1 has the edges $\{xa, xy, xz_1\}$, so K' has $\{xy, xz_1, xz_2\}$; thus xa is not an edge of K', and by Lemma 17 $a \notin K'$. Likewise, at y, $b \notin K'$. If w were in K', its K'-edges would be among wz_1, wz_2 (the others go to a, b), degree at most 2 in a 3-polytope, impossible. So $K' = \operatorname{conv}(x, y, z_1, z_2)$, a tetrahedron, and its edge $z_1 z_2$ is an edge of VF (Lemma 17, converse), which it is not. □

Theorem 13. *If every vertex lies in exactly two large 2-faces F_1 (n-gon) and F_2 (m-gon), and they lie in a common facet $D = \{n\} \times \{m\}$, then P is $\{n\} \times \{m\} \times \{\}$, $A_m \times \{n\}$ or $A_n \times \{m\}$.*

Proof. Let $e_1 = [a, b]$ and $e_2 = [c, d]$ be the VF-edges of F_1, F_2; then $\mathrm{VF}(D) = \operatorname{conv}(a, b, c, d)$. By Corollary 8 every VF-edge lies in $\operatorname{fan}(e_1)$ or $\operatorname{fan}(e_2)$, so every VF-vertex is a vertex of one of the two fans. The fan of F_1 is of type (i) with configuration $(m, q_2, \ldots, q_k)$, $x_1 = c$, $x_2 = d$, or of type (ii) with apexes $\{z_1, z_2\} = \{c, d\}$ (as $T = \mathrm{VF}(D)$); likewise for F_2 with n. The angle sum allows only $k = 3$, or $k = 4$ with configuration $(m, 3, 3, 3)$ (the configurations $(m, 4, p, p')$ and $(m, 3, p, p') \neq (m, 3, 3, 3)$ exceed 2π).

Simple-vertex step. A vertex z of $\operatorname{fan}(e_1)$ not in $\operatorname{fan}(e_2)$ has all its VF-edges in $\operatorname{fan}(e_1)$. If it has degree 4, each 3-subset of its edges spans a VF-facet (Lemma 18), and a VF-facet containing c and d contains the edge e_2 (Lemma 17), hence belongs to $\operatorname{fan}(e_2)$.

(i)/(i), $k = 3$. $z = x_3$ has the VF-edges za, zb, zc, zd in $\mathrm{fan}(e_1)$. If $z \notin \mathrm{fan}(e_2)$, the facet spanned by $\{za, zc, zd\}$ contains e_2, so $z \in \mathrm{fan}(e_2)$ after all. In a type-(i) fan of e_2 every vertex other than c, d is at distance $\sqrt{2}$ from c and d (Fact 11), so $\mathrm{ch}_{q_2} = |z - d| = \sqrt{2} = |z - c| = \mathrm{ch}_{q_3}$: the configuration is $(m, 4, 4)$. The other cases below leave only $(n, 4, 4)$ for F_2, with the same third vertex z. *Translate argument:* for a vertex t let D_t be the facet containing its two large 2-faces. For $t' \in D_t$, both large 2-faces of D_t at t' are large 2-faces of P, so $D_{t'} = D_t$. The $\{n\} \times \{4\}$ and $\{m\} \times \{4\}$ facets at the edge vz contain $F_1 + \tau$ and $F_2 + \tau$, $\tau = z - v \perp \operatorname{span} D$, so $D_z = D + \tau$. Hence the vertex set is $D_0 + X$ with X on a line, every point of X is a vertex, so $|X| = 2$ and $P = \{n\} \times \{m\} \times \{\}$.

(i)/(i), $k = 4$: both fans of type (i), that of F_1 with configuration $(m, 3, 3, 3)$. Here $|x_3 - d| = |x_4 - c| = \mathrm{ch}_3 = 1 \neq \sqrt{2}$, so neither x_3 nor x_4 lies in a type-(i) fan of e_2. Then both are simple with all edges in $\mathrm{fan}(e_1)$: x_3 has x_3a, x_3b, x_3d, x_3x_4 and x_4 has x_4a, x_4b, x_4x_3, x_4c. The facet G at x_3 spanned by $\{x_3a, x_3d, x_3x_4\}$ contains x_4 and a, hence the edge x_4a (Lemma 17); it does not contain b (else x_3b would be a fourth edge of G at the simple vertex x_3). So at x_4 its edges are $\{x_4a, x_4x_3, x_4c\}$, and G contains c and d, so $G \in \mathrm{fan}(e_2)$ and $x_3 \in \mathrm{fan}(e_2)$, a contradiction. Impossible.

(i)/(ii). $\mathrm{fan}(e_2)$ is the pair of $A_m \times \{\}$ with apexes a, b and trapezoid (c, d, y', x'), with $|dy'| = |y'x'| = |x'c| = 1$ and diagonals $\sqrt{1 + \mathrm{ch}_m} \in [1.7222, \sqrt{3})$, strictly between ch_5 and ch_6, hence not a chord ch_q of any polygon. If $k = 3$, x_3 must lie in $\mathrm{fan}(e_2)$ (as above) at distances ch_{q_2}, ch_{q_3} from d, c; but the vertices of $\mathrm{fan}(e_2)$ other than a, b, c, d are x', y', each at distance 1 from one of c, d and at the diagonal distance from the other, and the diagonal is not a chord; impossible. If $k = 4$, $(m, 3, 3, 3)$: by the argument of the previous case one of x_3, x_4 lies in $\mathrm{fan}(e_2)$; by symmetry say x_3; $|x_3 - d| = 1$ forces $x_3 = y'$. If $x_4 \notin \mathrm{fan}(e_2)$, x_4 is simple and its facet spanned by $\{x_4a, x_4x_3, x_4c\}$ meets the pyramid $\Pi(a; c, d, y', x')$ in a face containing a, c, y'; since c, y' are opposite vertices of the base, no proper face of the pyramid contains these three, so the facet is the pyramid and $x_4 \in \mathrm{fan}(e_2)$. Hence $\{x_3, x_4\} = \{y', x'\}$ and the neighbors of v are those of $\mathrm{fan}(e_2)$. *Cell partition:* Transitivity preserves fan types, even at $n = m$; denote the unique type-(ii) antiprism cell by C_t. For $t' \in C_t$ the m-gon of C_t through t' has C_t in its cycle, so it is not of type (i): it is the type-(ii) 2-face at t', and $C_{t'} = C_t$. The other neighbors a, b of v are the apexes of $\mathrm{fan}(e_2)$, so $a - v$ and $b - v$ are orthogonal to the 3-space of C_v and $C_a = C_v + (a - v)$. So $P = A_m \times \operatorname{conv} X$ with $\operatorname{conv} X$ a regular polygon, here F_1: $P = A_m \times \{n\}$. The case (ii)/(i) gives $A_n \times \{m\}$.

(ii)/(ii). The trapezoid vertex x of $\mathrm{fan}(e_1)$ (edges xa, xy, xc, xd in $\mathrm{fan}(e_1)$, the apexes being c, d) must lie in $\mathrm{fan}(e_2)$ (otherwise the facet spanned by $\{xy, xc, xd\}$ contains e_2), so $x \in \{x', y'\}$ at distance 1 or $\sqrt{1 + \mathrm{ch}_m}$ from c. But $|x - c| = \sqrt{2}$, since c is an apex of a prism over the A_n cell containing x. Impossible. □

Proof of the Main Theorem. Theorem 6 gives (K1)–(K3). Otherwise Proposition 10 leaves one large polygon, handled by Theorem 12, or two paired ones, handled by Theorem 13, both giving (K3). These constructions are convex and uniform. □

5 Computational verification

Exact algebra and certified intervals establish the finite assertions [14]. Checking notes give certificates, dependencies and commands.

With the two cited classifications and twelve geometric/computational premises as parameters, `Uniform5.Composition.mainClassification` proves

$$\forall P : \mathrm{Set}(\mathrm{Space}\,5), \quad \mathrm{UnitUniform}(5, P) \Longrightarrow K_1(P) \vee K_2(P) \vee K_3(P).$$

Here Space $5 = \mathbb{R}^5$; UnitUniform means a full-dimensional finite convex hull, vertex-transitive with recursively uniform facets and unit exposed edges. Formal K_1 uses Definition 1; K_2 allows every unit uniform 4-dimensional base, including both exceptional polytopes; K_3 uses Fact 2. Exposed segments are the polytope's edges. Congruence is intrinsic Euclidean congruence, with unit normalization matching the scaling convention. Lean proves the classification direction; construction uniformity and counts remain external.

Table 4 matches all twelve premise types to the mathematical conditions used above. Lean checks their conditional composition; the joint instantiation from the supplied certificates is external. The cited parameters are Facts 2–3. The logical axioms are standard; their list and exact declarations are in the checking notes.

Appendix A Supporting geometric calculations

These calculations justify the local geometric formulas and placement comparisons used above. Write V for the vertices of P, $\mathcal{L}_{16}$ for the finite facet list, $\mathcal{C}_{16}$ for its cycles, and $\mathcal{K}_{16}$ for the known constructions with polygon sizes at most 16. For a facet model K, X_K is its vertex figure, of radius r_K; the ambient vertex figure has radius $r = \sqrt{1 - 1/(4R^2)}$. The group E_K consists of exact center-fixing isometries of X_K preserving labeled ridges.

Lemma 14. *Let c be the centroid of V. Then all vertices lie on the sphere of some radius R about c, and c lies in the interior of P. For every face Φ of P the orthogonal projection o_Φ of c onto* $\operatorname{aff}\Phi$ *is the circumcenter of Φ, and lies in the relative interior of Φ. If Φ has circumradius R_Φ, then $|c - o_\Phi| = \sqrt{R^2 - R_\Phi^2}$.*

Proof. Every symmetry permutes V and so fixes c; by vertex-transitivity all vertices are equidistant from c. The centroid of the vertices of a full-dimensional polytope is an interior point. The vertices of Φ lie on the intersection of the sphere with $\operatorname{aff}\Phi$, a sphere in $\operatorname{aff}\Phi$ about o_Φ; they affinely span $\operatorname{aff}\Phi$, so o_Φ is the unique circumcenter of Φ. Since Φ is uniform, applying the first part to Φ shows that its circumcenter is its vertex centroid, which is relatively interior. The last formula is Pythagoras. □

Lemma 15. *Let P be a convex uniform d-polytope with unit edges and circumradius R, centered at 0, and v a vertex. The neighbors of v (the other ends of the edges at v) are exactly the vertices at distance 1 from v. They lie on the hyperplane $H = \{x : x \cdot v = R^2 - \frac{1}{2}\}$, every other vertex except v lies strictly on the side $x \cdot v < R^2 - \frac{1}{2}$, and* $\mathrm{VF} := H \cap P = \operatorname{conv}(\text{neighbors of } v)$ *is a $(d-1)$-polytope inscribed in a sphere of radius $r = \sqrt{1 - 1/(4R^2)}$ about $o_{\mathrm{VF}} = (1 - \frac{1}{2R^2})v$. The faces of* VF *correspond to the faces of P containing v (a j-face $\Phi \ni v$ gives the $(j-1)$-face $H \cap \Phi$). For a facet $K \ni v$ of circumradius R_K,* $\mathrm{VF}(K) := H \cap K$ *is inscribed in a sphere of radius $r_K = \sqrt{1 - 1/(4R_K^2)}$ in a hyperplane of H at distance $\sqrt{r^2 - r_K^2}$ from o_{VF}. An edge of* VF *coming from a p-gon has length $\mathrm{ch}_p = 2\cos(\pi/p)$. The point o_{VF} lies in the relative interior of* VF.

Proof. By Lemma 4 the vertices at distance 1 from v are neighbors, and conversely neighbors are at distance 1. For a vertex x, $|x - v|^2 = 2R^2 - 2x \cdot v$, so $|x - v| = 1$ iff $x \cdot v = R^2 - \frac{1}{2}$, and $|x - v| > 1$ (every other vertex $\neq v$) iff $x \cdot v < R^2 - \frac{1}{2}$. Hence H separates v from all non-neighbors, and $H \cap P$ is the convex hull of the neighbors; this is the standard vertex figure, whose face lattice is that of the faces through v. A neighbor x makes an angle θ with v at the center with $2R\sin(\theta/2) = 1$, so its distance from the axis $\mathbb{R}v$ is $R\sin\theta = \cos(\theta/2) = \sqrt{1 - 1/(4R^2)}$. The same computation

inside K gives r_K; the circumcenter of $\mathrm{VF}(K)$ is the projection of o_{VF}, which gives the distance. In a regular p-gon the two neighbors of a vertex are ch_p apart. Finally, write $o_{\mathrm{VF}} = t_0 v$ with $t_0 = 1 - 1/(2R^2)$. Since 0 is the centroid of the N vertices, $q := -v/(N-1)$ is the centroid of the vertices other than v, so $q \in P$, and $q \cdot v \le R^2 - \frac{1}{2}$, since every other vertex x has $x \cdot v \le R^2 - \frac{1}{2}$, with equality exactly for the neighbors. If some vertex other than v is not a neighbor, the inequality is strict, i.e. $-1/(N-1) < t_0$; also $t_0 < 1$. Then o_{VF} lies on the open segment from q to v, which passes through the interior point 0; every point of this open segment is interior to P (it lies strictly between the interior point 0 and a point of P, or is 0). Hence o_{VF} is an interior point of P lying in H, i.e. a relative interior point of $\mathrm{VF} = H \cap P$. If every vertex other than v is a neighbor (as in the 5-simplex), then VF is the convex hull of all other vertices, $t_0 = -1/(N-1)$ and $o_{\mathrm{VF}} = q$ is the centroid of the vertices of VF, which is a relative interior point of VF. □

Lemma 16 (Exact exclusions)**.** (Flat.) *A cycle with $\sum \alpha_i = 2\pi$ exactly is not the cycle of a 2-face of a convex 5-polytope.* (Degenerate.) *A root at $R = R_{\max}$ exactly does not give a convex uniform polytope.*

Proof. Lemma 5(a) and (d). □

Lemma 17 (Face lemma)**.** *Let X be a polytope, Φ a face of X and p, q vertices of Φ. If $[p, q]$ is an edge of X, it is an edge of Φ. Conversely every edge of Φ is an edge of X.*

Proof. $[p, q] \subseteq \Phi$ by convexity, and $[p, q] = [p, q] \cap \Phi$ is an intersection of two faces of X, hence a face of X contained in Φ, hence a face of Φ. The converse holds because a face of a face is a face. □

Lemma 18 (Simple vertices)**.** *A vertex x of degree 4 in a 4-polytope is simple: each facet containing x contains exactly three of the four edges at x, and each 3-subset of the edges at x is the set of edges at x of exactly one facet.*

Proof. The vertex figure of x is a 3-polytope with four vertices, a tetrahedron; its facets (all four triangles) correspond to the facets through x. □

Lemma 19 (Local uniqueness gives global uniqueness)**.** *Let S be the star of v_0 in some convex uniform 5-polytope, and suppose: for every convex uniform 5-polytope P with star S at v_0 and every neighbor w of v_0, the star of P at w is one and the same set Σ_w, independent of P. Then any two convex uniform 5-polytopes P_1, P_2 with star S at v_0 are equal.*

Proof. We show by induction on the edge distance from v_0 that P_1 and P_2 have the same vertices and the same stars there. Suppose P_1 and P_2 have the same star Σ at a vertex w. By vertex-transitivity of P_1 there is an isometry g_1 with $g_1(v_0) = w$ and $g_1(S) = \Sigma$. Then $g_1^{-1}(P_1)$ and $g_1^{-1}(P_2)$ are convex uniform 5-polytopes with star S at v_0; by hypothesis they have the same star at every neighbor u' of v_0. Applying g_1, P_1 and P_2 have the same star at every neighbor $u = g_1(u')$ of w (the neighbors of w are read off the common star Σ). The edge graph of a polytope is connected (Balinski [5]), and a polytope is the convex hull of its vertices, so $P_1 = P_2$. □

Lemma 20 (A word determines its angle)**.** *Let G be a p-gon of a uniform 4-polytope K (circumradius R_K), between cells A, B of circumradii R_A, R_B, and let $\rho = r_p$. Then the dihedral angle of K at G is*

$$\alpha = \arccos \sqrt{\frac{R_A^2 - \rho^2}{R_K^2 - \rho^2}} + \arccos \sqrt{\frac{R_B^2 - \rho^2}{R_K^2 - \rho^2}}.$$

In particular the word (K, p, A, B) determines α, even when its polygon occurrences lie in different orbits of $\mathrm{Sym}(K)$.

Proof. In the plane through the center f of G perpendicular to G, the two cells give rays u_A, u_B bounding a convex wedge of angle α, and the center O of K projects strictly inside it (Lemma 14 applied to K, and Lemma 5(b),(c) one dimension down). With $D = |O - f|^2 = R_K^2 - \rho^2$ we have $(O - f) \cdot u_A = \sqrt{R_A^2 - \rho^2} > 0$ and likewise for B, since the projection of O on each cell is its relatively interior circumcenter. The direction of $O - f$ lies between the rays, so the two angles to it add to α. □

Corollary 21. *Let X be any convex uniform 5-polytope with unit edges and circumradius R_X, and G a 2-face of X all of whose facets (around G) belong to $\mathcal{L}_{16}$, with G at most 16-gonal. Then R_X lies in a root interval of the cycle of G, and that interval belongs to Ω.*

Proof. The cycle of G is a library cycle (Lemma 5(a)), hence in $\mathcal{C}_{16}$; it is not flat (Lemma 16); $1/R_X$ is a zero of g in $[0, 1/R_{\max})$ by Lemma 5(c),(d), so it lies in a root interval, which is not a degenerate one (Lemma 16). □

Lemma 22. *All cycles of the 2-faces of P at a vertex v lie in one radius class, the class containing R, and none of them is removed by the closure filter.*

Proof. By Corollary 21 each of these cycles has a root interval in Ω containing R, so all these intervals lie in the class of R. Let K be a facet of P at v and (p, C_a, C_b) a polygon orbit of K. Since K is vertex-transitive, the orbit has a member $G \ni v$, a 2-face of P whose cycle contains K between C_a and C_b; this cycle is in the class of R. So the set of cycles of P's 2-faces at v is never touched by the filter (by induction over the iterations). □

Lemma 23 (Exact equivalence of placements)**.** *Let g be an exact isometry of the model vertex figure X_K fixing its center, preserving its labeled ridges and VF-edge polygon sizes, and mapping an ordered source ridge correspondence o to another one o' (that is, $g(o_i) = o'_i$). Then for every admissible R in the class ($R > R_K$) the facets placed over a given target ridge by o and by o', with the hinge on the side opposite to the existing facet, coincide as labeled point sets.*

Proof. g fixes the template center, so it preserves the distance of the source ridge from it and carries its outward normal to that of o'. With the hinge on the opposite side, the new facet's normal is computed from $d_K(R)$, that distance and the target ridge alone, so it is the same for both options. If A_o, $A_{o'}$ are the two gluing maps, $A_{o'} \circ g$ and A_o have the same ordered ridge images and the same outward normal; they agree on three non-collinear ridge vertices and on the facet center, which is not in their affine plane, and so they are equal on the facet hyperplane. Hence the placed facets coincide as functions of R, whether or not g reverses orientation, and g preserves the adjacent labels used by later filters. □

Lemma 24 (Congruence from discrete data)**.** *Let X, X' be convex uniform 5-polytopes with the same circumradius R and vertex figures V, V' indexed by the same set, with the same facets: the same index sets, the same types, and correspondences that agree modulo E_K. Then a linear isometry A with $AV_i = V'_i$ for all i exists.*

Proof. (1) Take a facet f with $V[S_f] = g(X_K)$ and $V'[S_f] = g'(X_K)$ and $c' = c \circ e$ with $e \in E_K$ realized by the isometry ε. Then $V[c(t)] \mapsto V'[c(t)]$ is $g'\varepsilon^{-1}g^{-1}$, an isometry. Both facet sets lie in hyperplanes at the same distance $\sqrt{r^2 - r_K^2}$ from the center O, so it extends to a linear isometry A of $\mathbb{R}^4$. (2) Let f' share a ridge ρ (at least three non-collinear points) with f. Project to the 2-plane W orthogonal to the directions of ρ: ρ becomes a point p, and the hyperplanes through ρ become lines through p, tangent at their feet $T = dn$ to circles about O. The center O is interior to the vertex figure (Lemma 15), so it lies in the open wedge at p bounded by the two facet lines (angle below π). Each facet's circumcenter lies on its ray; it is relatively interior to the facet (the facet is the vertex figure of a uniform 4-polytope), so $T \neq p$. Hence the two feet lie strictly on opposite sides of the line pO, and the hyperplane of f' is the tangent line from p at distance $d_{K_{f'}}$ on the side opposite to f's foot; it is the same for AV and V'. (3) Inside that hyperplane $AV[S_{f'}]$ and $V'[S_{f'}]$ are related by an index-preserving isometry, as in (1), which fixes ρ pointwise and the common circumcenter, which is not in the plane of ρ; so it is the identity. (4) The facet graph of V through ridges is connected, so $AV = V'$. □

Lemma 25 (Coset comparison)**.** *Let S be the full star at v_0 of a hypothetical convex uniform polytope of circumradius R, centered at the origin, and let T be a group of exact linear isometries fixing S. At a neighbor w, let $g_1, \dots, g_k$ be a complete, nonempty list of candidate linear isometries for its star, with a true congruence g_i among them. Suppose that for every j there is $\tau_j \in T$ and a certified Frobenius-norm bound $\|g_1\tau_j - g_j\|_F \leq \varepsilon_j$, and $2R\max_j \varepsilon_j < 1/2$. Suppose each g_j has anchor vertices a in S spanning $\mathbb{R}^5$, and every image $g_j(a)$ belongs to a known facet through w. Then every candidate gives the same full star at w.*

Proof. The identity

$$g_i\tau_i^{-1}\tau_j - g_j = (g_i - g_1\tau_i)\tau_i^{-1}\tau_j + (g_1\tau_j - g_j)$$

bounds the operator norm by $\varepsilon_i + \varepsilon_j$. On an anchor of norm R, the two images are less than $1/2$ apart. Both are vertices of the hypothetical polytope: one lies in the true star $g_i(S)$, the other in the known facet. By Lemma 4 they coincide. The spanning anchors determine the isometry, so $g_j = g_i\tau_i^{-1}\tau_j$ and $g_j(S) = g_i(S)$. □

Lemma 26 (Rooted determinism)**.** *Fix a common circumcenter c and an admissible radius R (a radius in the class interval with $R > R_K$ for every facet type K used). A* rooted encoding *of a star at v_0 records the full template type of an initial facet and its root frame (the images of the template's base vertex and its neighbors), and, along a connected traversal of the facet adjacency graph of the vertex figure (facets adjacent through cells containing v_0), the type of each further facet and the ordered correspondence of its model neighbor indices, taken modulo the exact stabilizer of the base vertex in the* full *template. If two stars at the same R have equal rooted encodings in root frames whose identification also identifies c, they are equal as sets of full facets.*

Proof. The root frame fixes the template map on the base vertex and its neighbors, which affinely span the facet hyperplane, so it fixes the full root facet. Suppose a facet F is fixed and $C \ni v_0$ is a cell of F at which the traversal reaches the next facet F' of type K'. The affine hull of C is fixed. The center of F' is at distance $\sqrt{R^2 - R_{K'}^2} > 0$ from the fixed common center c, so the hyperplane of F' is one of the two hyperplanes through $\operatorname{aff} C$ at that distance, and, as in step (2) of the proof of Lemma 24 one dimension up (the center of the polytope is interior, the facet centers are relatively interior by Lemma 14), it is the one whose foot lies on the side opposite to that of F. The ordered correspondence fixes the images of v_0 and of the vertex-figure points of C; together

with the center of F', which is not in aff C, they affinely span the hyperplane of F', so the isometric template map, and hence the full facet F', is unique. Changing a correspondence by an element of the full-template base stabilizer does not change the placed full facet. The facet graph of the vertex figure is connected, so induction reaches every facet. If the stars are given as functions of R on an interval, the conclusion is equality of the placement functions at each common admissible R; the unique-root step then selects one value. □

Lemma 27 (F5: cycle types)**.** *Let F be a 2-face of P with $n \notin \{3,4,5,6,8,10\}$ sides. Then the cycle of F is one of*

(i) $\{n\}\times\{q_1\},\dots,\{n\}\times\{q_k\}$ *(all cells P_n), with $k \in \{3,4,5\}$ and $\sum_i(1-2/q_i) < 2$;*

(ii) $A_n\times\{\}$, $A_n\times\{\}$, $\{n\}\times\{q\}$*;*

(iii) $A_n\times\{\}$, $A_n\times\{\}$, $\{n\}\times\{q_1\}$, $\{n\}\times\{q_2\}$ *with $\{q_1,q_2\} \in \{\{3,3\},\{3,4\},\{3,5\}\}$.*

Proof. By Facts 2–3 the cells at F are P_n or A_n and the facets $\{n\}\times\{q\}$ or $A_n\times\{\}$. The face angles of the tangent cone at F (Lemma 5) are $\pi(1-2/q)$ for $\{n\}\times\{q\}$ (between two P_n) and $\pi/2$ for $A_n\times\{\}$ (between A_n and P_n), with sum below 2π. Both neighbors of an A_n ray are $A_n\times\{\}$, each containing one A_n cell at F, so j rays of type A_n cost $2j$ distinct right angles, and $j \le 1$. If $j = 0$ we get (i), with $k \ge 3$ and $k \le 5$ since each term is at least $1/3$. If $j = 1$ there are $m \ge 0$ duoprisms between the two P_n rays, and the cycle has $m+2$ rays. The tangent cone at F is a full-dimensional pointed cone in the 3-dimensional space N, so a section of it by a positive affine functional is a convex polygon and the cone has at least three extreme rays; hence $m = 0$ is impossible. $m = 1$ is (ii); $m = 2$ needs $1/q_1 + 1/q_2 > 1/2$, i.e. (iii); $m \ge 3$ gives a sum of at least $\pi + 3\pi/3$, too large. □

Type (i). The cells at F are $F\times[v, v+u_i]$ with unit $u_i \perp L$; consecutive u_i, u_{i+1} span the q_i-gon of $\{n\}\times\{q_i\}$ at v, so $|u_i - u_{i+1}| = \mathrm{ch}_{q_i}$. All u_i satisfy $u_i \cdot W = 1/2$, so they lie on a circle of radius $S = \sin\alpha$ on the cone of half-angle α about W, $\cos\alpha = 1/(2|W|)$. By Lemma 5(c), W is strictly inside the cone spanned by the u_i, i.e. the circle's center is strictly inside the inscribed polygon of the u_i: all central angles $2\arcsin(\cos(\pi/q_i)/S)$ are below π and they sum to 2π. Let $f_c(S) = \sum_i 2\arcsin(\cos(\pi/q_i)/S)$ on $[c_{\max}, 1]$, $c_{\max} = \max_i \cos(\pi/q_i)$; f_c is strictly decreasing and $f_c(1) = \sum \pi(1-2/q_i) < 2\pi$. Hence:

Lemma 28 (Type (i) radius and closability)**.** *A configuration $c = (q_1,\dots,q_k)$ occurs in a type-(i) cycle only if it is* closable*: $f_c(c_{\max}) > 2\pi$. Then $f_c(S) = 2\pi$ has a unique solution $S \in (c_{\max}, 1)$, the* configuration radius $R_c = 1/(2\sqrt{1-S^2})$ *depends only on the multiset c, and*

$$R^2 = r_n^2 + R_c^2.$$

*Moreover $R_c \ge r_{q_{\max}}$. For $k = 3$, closability is equivalent to the triangle with sides $\mathrm{ch}_{q_1}, \mathrm{ch}_{q_2}, \mathrm{ch}_{q_3}$ being acute; for $k = 4$ to the circle's center being strictly inside the inscribed quadrilateral. If $f_c(c_{\max}) = 2\pi$ exactly (*degenerate*), the center lies on the hyperplane of a facet, which is impossible.*

Proof. All but the formula $R_c \ge r_{q_{\max}}$ and the triangle criterion is above. $S \ge \cos(\pi/q_{\max})$ gives $R_c^2 = 1/(4(1-S^2)) \ge 1/(4\sin^2(\pi/q_{\max}))$. For three rays: the foot of the perpendicular from 0 to the plane of $u_1u_2u_3$ is the circumcenter o_T of the chord triangle, and $o_T \cdot u_i = |o_T|^2$ for each i, so $W = o_T/(2|o_T|^2)$ (both satisfy the three equations $x \cdot u_i = 1/2$, which have a unique solution); so W is interior to the cone iff o_T is interior to the triangle iff the triangle is acute, and a right triangle puts W in the plane of a facet. The same argument with the circle gives the $k = 4$ form. □

Lemma 29 (closed forms)**.** *Let $n \geq 17$, $\theta = \pi/n$, $c = \cos\theta$ and $x = c^2$. Then*

$$\delta_{(4,4)}(n) = \tfrac{1}{4}, \qquad \delta_{(3,6)}(n) = \frac{x}{4x-1} = \frac{1}{4-\sec^2\theta}, \qquad \delta_{(3,3,3)}(n) = \frac{2c+1}{8(1+c)} = \frac{h_A(n)^2}{4}.$$

$\delta_{(3,6)}$ decreases and $\delta_{(3,3,3)}$ increases with n, so for $n \geq 17$, including the limit $n = \infty$, $\delta_{(3,6)}(n) \in [1/3,\ \delta_{(3,6)}(17)]$ with $\delta_{(3,6)}(17) = \cos^2(\pi/17)/(4\cos^2(\pi/17) - 1) = 0.3372617\ldots < 0.337262$, and $\delta_{(3,3,3)}(n) \in [h_A(17)^2/4,\ 3/16]$ with $h_A(17)^2/4 = 0.1869633\ldots$; the value $3/16$ is attained only in the limit $n = \infty$.

Proof. For a chord triangle with squared sides A, B, C the circumradius satisfies $S^2 = ABC/((A+B+C)^2 - 2(A^2+B^2+C^2))$, and $R_c^2 = 1/(4(1-S^2))$, $r_n^2 = 1/(4(1-x))$. For $(n,3,6)$ the sides are $2c, 1, \sqrt{3}$, which gives $S^2 = 3x/(-1+8x-4x^2)$ and $\delta_{(3,6)} = x/(4x-1)$; the triangle is acute since $1/2 < x < 1$, so this is the closable solution. For $(n,4,4)$ the sides are $2c, \sqrt{2}, \sqrt{2}$, giving $S^2 = 1/(2-x)$ and $\delta = 1/4$ (the prism P_n); this triangle is also acute. For $(n,3,3,3)$ put $S^2 = 1/(3-2c)$ and $\sin\alpha = 1/(2S)$; then $\pi/6 < \alpha < \pi/4$, the three unit chords subtend 2α each, the remaining arc $2\beta = 2\pi - 6\alpha$ lies in $(\pi/2, \pi)$, and $\sin\beta = \sin 3\alpha = \sin\alpha(3 - 4\sin^2\alpha) = 2c\sin\alpha$, so its chord is $2S\sin\beta = 2c$: all four arcs are below π and sum to 2π. Substitution gives $\delta_{(3,3,3)} = (2c+1)/(8(1+c)) = \big(1-1/(2(1+c))\big)/4 = h_A(n)^2/4$ (the antiprism A_n), using $\cos^2(\theta/2) = (1+c)/2$. Finally $d\delta_{(3,6)}/dc = -2c/(4c^2-1)^2 < 0$ and $d\delta_{(3,3,3)}/dc = 1/(8(1+c)^2) > 0$, and c increases to 1 with n. □

For any configuration tail b, write $\delta_b(n) = R^2_{(n,b)} - r_n^2$; h_A is the antiprism height defined in Section 4. The seven exceptional configurations are $(m,3,7)$ for $m = 38, 39, 40, 41$, $(23,3,8)$, $(17,3,9)$ and $(19,4,5)$.

Lemma 30 (Partner-radius filter)**.** *If the fan of F (an n-gon) is paired of type (i) with configuration (m, c_1), c_1 one of $(3,7)$, $(3,8)$, $(3,9)$, $(4,5)$, then the fan of the partner m-gon is of type (i) with configuration (n, c_2) and $\delta_{c_1}(m) = \delta_{c_2}(n)$, or of type (ii) with $\delta_{c_1}(m) = h_A(m)^2/4$. For the seven exceptional fans this forces $c_2 = c_1$ and $n = m$.*

Proof. From F: $R^2 = r_n^2 + R^2_{(m,c_1)} = r_n^2 + r_m^2 + \delta_{c_1}(m)$. The partner's cycle contains $\{m\} \times \{n\}$, so it is of type (i) with a configuration (n, c_2) or of type (ii) with $q = n$ (type (iii) needs parameters at most 5); this gives $R^2 = r_m^2 + r_n^2 + \delta_{c_2}(n)$ or $R^2 = R^2_{A_m} + r_n^2 = r_m^2 + h_A(m)^2/4 + r_n^2$. For the seven fans $\delta_{c_1}(m) > 99$ (from 99.67 for $(17,3,9)$ to 1040.7 for $(41,3,7)$), while $h_A^2/4 < 1/4$, $\delta_{(4,4)} = 1/4$, $\delta_{(3,3,3)} < 3/16$, $\delta_{(3,6)} \leq 0.3373$, so c_2 is one of the bounded families, and a certified comparison of the finitely many values in the checking notes leaves only $c_2 = c_1$, $n = m$. □

Appendix B Constructions and counts

For the control table, $\mathcal{H}_{10} = \mathcal{L}_{10}$ and $\mathcal{H}_{16} = \mathcal{L}_{16}$. The small control $\mathcal{H}_6$ also retains the fixed eight- and ten-gonal facets; P_{ctl} is its family-parameter cutoff. These are control libraries, not additional families of the theorem. The following tables record the finite enumeration and fix the naming convention for the constructions. Detailed certificate statements and recorded controls are in the checking notes.

Counting the Wythoffian constructions (Definition 1) and prisms over the four-dimensional bases as listed in Möller [17, Sections 1.2 and 9.1; Table 8.7, pp. 113–114] gives

$$19 + 31 + 8 + 47 - 1 = 104, \qquad 47 = 45 + 2.$$

Table 1: Counts of facet types, cycles, radius classes and vertex figures at polygon cutoffs 6, 10 and 16. A surviving signature need not be realizable.

	$\mathcal{H}_6$	$\mathcal{H}_{10} = \mathcal{L}_{10}$	$\mathcal{H}_{16} = \mathcal{L}_{16}$
canonical facet types	76	106	181
cycles	2,559	5,471	14,965
flat (exact) / degenerate roots (exact)	790 / 23	1,703 / 39	7,080 / 57
radius classes / surviving the filter	760 / 136	1,790 / 278	4,224 / 563
surviving classes with a known radius	135	277	562
vertex-figure signatures	168	332	629
distinct known polytopes	167	331	628

Table 2: Coxeter diagrams and node orders used in the names.

Group	Diagram (nodes $0, 1, \dots$)
A_4, A_5	linear, all branches 3
B_4	linear $3, 3, 4$; the 4 between nodes 2 and 3
B_5	linear $3, 3, 3, 4$; the 4 between nodes 3 and 4
D_4	node 1 joined to nodes 0, 2, 3 (all 3)
D_5	node 2 joined to nodes 1, 3, 4; node 0 joined to node 1
F_4	linear $3, 4, 3$; the 4 between nodes 1 and 2
H_4	linear $5, 3, 3$; the 5 between nodes 0 and 1

The 47 prisms include the tesseract prism, which is the B_5 5-cube; the subtraction counts it once. This is a finite-sublist count, not a disjoint count after every overlap with the infinite families. The snub 24-cell is Coxeter's construction [9, Section 8.4]; Conway and Guy give the grand antiprism [8]. Further historical sources on semiregular polytopes are Gosset, G. Blind and R. Blind, and Elte [12, 6, 11]. The construction sources include Coxeter and Johnson [9, 10, 15]; Schulte cites Johnson's unpublished manuscript [16] for the four-dimensional list; it was not available to us. Catalogs and problem records are [3, 4, 19, 20]; the broader uniform-polyhedron lists, including the convex subclass, are treated by [10, 23, 22]; see Grünbaum [13] for the convex terminology. Càllisto's honeycomb work supplies a methodological precedent for combining Lean arguments with external finite checks [7, Sections 1 and 11, pp. 4 and 31–32]; none of its theorems is a premise here.

Appendix C Finite certificates

The following statements give the exact finite and interval assertions used in the main proof. The certificates concern congruence, facet geometry, cyclic closure, vertex figures, radii and fan distances. Parenthetical short names identify the corresponding certificate contracts. Write $g(s) = \sum_i \theta_i(s) - 2\pi$ for the closure function in reciprocal radius $s = 1/R$. A radius class is an overlap component of root intervals, with its interval hull. Library geometry includes the cited construction formulas and exact incidence correspondence; the checking notes give the algorithms and all recorded checks.

Theorem 31 (Case B)**.** *Every convex uniform 5-polytope with a 2-face of at least 17 sides is congruent to $\{n\} \times Q$ with Q a convex uniform polyhedron, or to $\{q\} \times A_n$, for integers $n \geq 17$ and $q \geq 3$.*

The proof is Section 4: the fan reduction and the two product theorems.

Table 3: The 33 aliases of the facet library and their canonical names (Computation 34).

Canonical	Aliases	Canonical	Aliases
A40001	A41000	B41000	D40001, D40010, D41000
A40010	A40100	B41010	D41011, F40010, F40100
A40011	A41100	B41100	D40101, D40110, D41100
A40101	A41010	B41110	D41111, F40011, F41100
A40111	A41110	F40101	F41010
A41011	A41101	F40111	F41110
B40001	pr-cube, duoprism-4-4	F41011	F41101
B40010	D40011, D41001, D41010	pr-oct	antiprism-prism-3
B40100	D40100, F40001, F41000		
B40110	D40111, D41101, D41110		

Corollary 32 (Decision rule)**.** *Let x, y be two vertices of one convex uniform polytope with unit edges (possibly hypothetical), and let $[\ell, u]$ be a rigorous enclosure of $|x-y|^2$. Then:*

(a) *if $\ell \le 0$ and $u < 1$, then $x = y$;*

(b) *if $0 < \ell$ and $u < 1$, we have a contradiction (the configuration does not occur);*

(c) *if $0 < \ell$ (and $u \ge 1$), then $x \ne y$;*

(d) *if $\ell \le 0$ and $u \ge 1$, nothing is concluded: $x = y$ and $x \ne y$ both remain possible.*

For $3 \le P \le 16$ let $\mathcal{L}_P$ be the following list of names of convex uniform 4-polytopes all of whose 2-faces have at most P sides. For $P = 16$ it has 214 names:

- the Wythoffians `A4`*abcd*, `B4`*abcd*, `D4`*abcd*, `F4`*abcd*, `H4`*abcd* (*abcd* $\in \{0,1\}^4 \setminus \{0000\}$; 75 names; their 2-faces have at most 10 sides);
- `sadi` (snub 24-cell) and `gap` (grand antiprism);
- `pr-`X, the prism over X, for the 18 Platonic and Archimedean solids X;
- `antiprism-prism-`$n = A_n \times \{\}$, $3 \le n \le 16$ (14 names);
- `duoprism-`p`-`$q = \{p\} \times \{q\}$, $3 \le p \le q \le 16$ (105 names; the prisms $P_q \times \{\}$ are `duoprism-4-`q).

Names, diagrams and cell labels are listed in Appendix B. For smaller P, the definition of $\mathcal{L}_P$ retains exactly the names in this list whose every 2-face has at most P sides.

The small control. The $P = 6$ quick control $\mathcal{H}_6$ retains fixed 8- and 10-gon facets and has 136 classes rather than the literal cutoff's 129, whereas $\mathcal{H}_{10} = \mathcal{L}_{10}$ and $\mathcal{H}_{16} = \mathcal{L}_{16}$; we write P_{ctl} for its family-parameter bound.

Lemma 33 (Library completeness)**.** *Every convex uniform 4-polytope all of whose 2-faces have at most 16 sides is congruent to the polytope of some name in $\mathcal{L}_{16}$.*

Certified computation 34 (MERGE: exact congruence of the 33 aliases)**.** Each of the 33 aliases is congruent to its canonical name.

Certified computation 35 (LIB: facet data)**.** For each of the 181 canonical types, the circumradius and the dihedral angles of all polygon orbits are enclosed in balls, and the model is constructed with all coordinates enclosed in balls.

Certified computation 36 (LINK: certified stage-1 input)**.** For each of the 181 canonical types (assuming the construction properties of Section 1), the list of polygon occurrences at the base vertex, with their two cells and labels, is complete and correct (1,277 occurrences, 558 distinct words (K, p, A, B)); the exported circumradius encloses R_K and each exported angle encloses the angle of Lemma 20. Every radius and angle entry of the stage-1 input tables at $P_{\mathrm{ctl}} = 6$, 10 and 16 contains these rigorous enclosures, and each table lists exactly the words of its corresponding control library $\mathcal{H}_P$.

Certified computation 37 (EXACT: certified exact exclusion inputs)**.** The squared circumradii of the 181 canonical types and of the 44 named cell types and, for each of the 558 incident words, the algebraic cosine used by the exclusions, are exactly the values of the constructions of Section 1; each cosine equals the displayed expression with positive square roots.

Write $R_{\max} = \max_i R_{K_i}$. The closure filter removes cycles lacking a required facet incidence, as in Section 3; retained leaves are the interval subdivisions forming the root cover.

Certified computation 38 (S1: stage 1)**.** (i) The list $\mathcal{C}_{16}$ of library cycles, up to rotation and reversal, has 14,965 elements, and it is complete.

(ii) For each cycle the zeros of its closure function g on $[0, 1/R_{\max}]$ (with the domain extended to the upper end of the ball enclosing $1/R_{\max}$) lie in a listed finite set of *root intervals*, formed by merging retained leaves of width below 10^{-20} in s; this threshold does not bound the width of a merged interval. Of the cycles, 14,428 have at least one root interval, 537 have none, and none is undecided.

(iii) Exactly 7,080 cycles have $\sum \alpha_i = 2\pi$ exactly (flat), and exactly 57 root intervals are the root $R = R_{\max}$ exactly (degenerate); each is proved by exact algebra.

Definition 39. Ω is the set of root intervals of all cycles in $\mathcal{C}_{16}$ other than the flat cycles and the degenerate roots: 7,291 intervals, all bounded, with $\min \Omega = 0.6454972\ldots$ (the 5-simplex) and $\max \Omega = 63.2195709614\ldots$.

Certified computation 40 (FILTER)**.** The intervals of Ω form 4,224 radius classes, of which 563 survive the closure filter.

Let I be a surviving class with interval $[R_{\mathrm{lo}}, R_{\mathrm{hi}}]$. By Lemmas 15 and 22, if $R \in I$ then the vertex figure VF of P at v satisfies:

(a) VF is a convex 4-polytope inscribed in the 3-sphere of radius $r = \sqrt{1 - 1/(4R^2)}$; each facet is $\mathrm{VF}(K)$ for a facet type K that occurs in a cycle of I, lying in a hyperplane at distance $\sqrt{r^2 - r_K^2}$ from the center, with the center on its inner side;

(b) every ridge (2-face) of VF is the vertex figure of a cell and lies in exactly two facets of VF, with the same cell label;

(c) around every edge of VF (a polygon of P at v) the facets and ridge labels, read cyclically, form a cycle of I;

(d) distinct vertices of VF are at distance at least 1 (Lemma 4).

A *signature* of a vertex figure is its multiset of facet types together with the multiset of canonical cyclic words around its edges.

Certified computation 41 (S2: stage 2)**.** For each of the 563 surviving classes, an exhaustive search, sound for every R in the class interval, outputs a finite list of complete candidate encodings; their signatures are 629 in total (one class, the class at $R = 1.0505014948\ldots$ formed by a cycle of three 5-cells and a 16-cell around a triangle, has none). For every R in the class and every vertex figure satisfying (a)–(d), its signature is one of the signatures listed for the class. This is a coverage statement: retaining an encoding does not prove that all its assembly equations have a simultaneous exact solution. Existence of the known models is supplied separately by Section 1 and their named constructions.

Certified computation 42 (RIDGES: certified ridge lists)**.** For each of the 181 canonical types K, the list of 2-faces of the model vertex figure X_K (which point sets, their cyclic order, their cell labels) is correct.

Certified computation 43 (ROOT: unique radius)**.** For every one of the 629 vertex-figure signatures, one of its edge cycles has a closure function g with exactly one zero on the *whole* class interval $[1/R_{\mathrm{hi}}, 1/R_{\mathrm{lo}}]$ of the surviving stage-1 class: g has certified nonzero values of opposite signs at the two ends, and g', enclosed over the whole interval, excludes 0.

The extra encoding has eight neighbors, circumradius $R = 1$, and the facet multiset of two 5-cells, six triangular duoprisms and eight tetrahedral prisms (Section 3). Its full coordinates and certificates are in the checking notes.

Certified computation 44 (DEV: the extra vertex figure is refuted)**.** No convex uniform 5-polytope has an actual star represented by this extra candidate encoding in its class interval.

A signature has verdict `KNOWN` when identified with a named known model and `REFUTED` when every represented actual star is excluded.

Proposition 45 (stage 3, proof route)**.** *Let P be a convex uniform 5-polytope with all 2-faces at most 16-gonal, and let σ be the signature of its vertex figure at v_0, in the surviving class I containing R. If σ has verdict `KNOWN` with name Q, then $P \cong Q$; if σ has verdict `REFUTED`, P does not exist.*

Certified computation 46 (B2)**.** Let $n \geq 17$. None of the following radii lies in an interval of Ω (Definition 39):

- $\sqrt{r_n^2 + R_c^2}$ for every closable configuration c (any entries ≥ 3, $k \in \{3,4,5\}$, $\sum(1 - 2/q_i) < 2$);
- $\sqrt{R_{A_n}^2 + r_q^2}$ for every $q \geq 3$;
- $\sqrt{R_{A_n}^2 + 1/(2(1+\gamma))}$ for every root $\gamma \in (-1, 1)$ of the type-(iii) polynomial equations, geometric or not.

Certified computation 47 (P1)**.** Of the 168 cyclic configurations with entries in 3..16 and angle sum below 2π, 137 are closable, 28 not closable and 3 degenerate. Of the 137, 83 contain a family polygon. For 67 of these 83 there is a family polygon q in c such that $X := R_c^2 - r_q^2$ is certified different from $h_A(q)^2/4$, from $1/4$ ($c'' = (4,4)$), from the range $[0.186963, 3/16]$ of $\delta_{(3,3,3)}$, from the range $[1/3, 0.337262]$ of $\delta_{(3,6)}$, and from each of the finitely many values $\delta_{(3,7)}(n)$, $n \leq 41$, $\delta_{(3,8)}(n)$, $n \leq 23$, $\delta_{(3,9)}(17)$, $\delta_{(4,5)}(n)$, $n \leq 19$. The 16 others are $(4,4,q)$ and $(3,3,3,q)$, $q \in \{7,9,11,\ldots,16\}$.

A point p is *admissible* for a fan if it lies on the VF sphere ($|p-v| = 1$, $|p-c| = R$), on the closed inner side of every fan facet's hyperplane, and at distance at least 1 from every fan vertex. Every VF-vertex that is not a fan vertex is admissible (VF is convex and the fan facets are VF-facets; Lemma 4). An *allowed* fan vertex is a fan vertex other than a, b and, for a paired fan, the partner ends. The partner ends are distinct from the other fan vertices: in type (ii) the apexes satisfy $z_j - v \perp M$ while $x - v, y - v \in M$; in type (i) the x_i are distinct lateral edges.

Certified computation 48 (W)**.** For each fan in the list below and every real $n \geq 17$, including the limit $n = \infty$ (and, in (c), every real $m \geq 17$ including $m = \infty$; in (d), every *integer* m in the stated range), there are no two points p, p' with $|p - p'| \geq \mathrm{ch}_{17}$, each admissible or an allowed fan vertex, other than two fan vertices. The list:

(a) type (i): the 70 closable configurations with entries at most 16 that are not excluded by (P1);

(b) type (ii), $q = 3, \ldots, 16$; type (iii), $(3, 3)$, $(3, 4)$, $(3, 5)$;

(c) paired: $(m, 4, 4)$, $(m, 3, 3, 3)$, $(m, 3, 6)$ and type (ii) with $q = m$, all $m \geq 17$ and $m = \infty$;

(d) paired, bounded, with m an integer: $(m, 3, 7)$, $17 \leq m \leq 37$; $(m, 3, 8)$, $17 \leq m \leq 22$; $(m, 4, 5)$, $17 \leq m \leq 18$, for all real $n \geq 17$; and $(m, 3, 7)$, $38 \leq m \leq 41$, $(23, 3, 8)$, $(17, 3, 9)$, $(19, 4, 5)$ at the integer point $n = m$ only.

Moreover, for each fan in the list, every pair of fan vertices other than e, the partner edge and the diagonals $x_i x_j$ of a type-(i) fan has distance below ch_{17} (the largest is the 16-gon chord $1.961571\ldots$; in type (iii) $|z_1 - z_2| \leq 1.943206$).

Table 4: The twelve formal premises and their mathematical correspondence. Exact declarations are in the checking notes.

Premise	Mathematical condition and location
Library geometry	The model hulls, vertices, cell incidences, radii and angles have the stated geometric meanings (Computations 35, 36, 37 and the construction definitions).
Cycle coverage	Every possible facet cycle is represented and every closure root lies in the reported intervals (Computation 38).
Vertex-figure coverage	Every actual vertex figure at every admissible radius has a reported signature (Computation 41).
Unique radius	Each surviving signature has one scalar root on its entire class interval (Computation 43).
Extra-case refutation	No actual star realizing the extra candidate can occur (Computation 44).
Radius separation	No large-polygon radius belongs to the small-cycle cover (Computation 46).
Configuration bounds	The excluded unpaired configurations violate the exact radius comparisons (Computation 47).
Fan distances	All admissible and allowed point pairs satisfy the stated long-pair exclusion (Computation 48).
Partner radii	The exceptional paired cases have equal polygon parameters (Lemma 30).

Premise	Mathematical condition and location
Full-star coverage	Every actual geometric star is represented by a candidate encoding, with correct correspondences (Section 3, Proposition 45, and Lemma 26).
Identification	Every represented admitted star equals the star of its separately constructed known polytope; remaining candidates are excluded (same argument and Proposition 45).
Neighbor forcing	Every possible continuation at a neighbor has the same full star (Proposition 45; Lemma 19 propagates this globally).

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