A planar graph without four- or nine-cycles that is not three-colorable

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Abstract

Lu, Wang, Wang, Bu, Montassier and Raspaud asked which cycle lengths, together with length four, force proper three-colorability of planar graphs, as recorded by Kang, Jin and Wang. Cohen-Addad, Hebdige, Král', Li and Salgado disproved Steinberg's conjecture for lengths four and five. Kostochka and Yancey proved a sharp general lower bound on the number of edges of a four-critical graph. We give an 82-vertex planar graph without cycles of length four or nine that is four-critical and attains that bound, answering the length-nine case of that question. The graph is obtained by Dirac–Hajós–Gallai–Ore compositions of complete four-vertex graphs. Its non-three-colorability follows by induction on this construction; a rotation system, cycle enumeration and three-colorings after edge deletion establish the remaining properties. Lean verifies the rotation certificate, the exclusion of four- and nine-cycles, and non-three-colorability of the same graph.

1 Introduction

All graphs in this paper are finite and simple. A proper three-coloring is a map $f : V(G) \to \{0, 1, 2\}$ with $f(u) \neq f(v)$ for every edge uv. A cycle is simple; it need not be induced. Thus forbidding a cycle of length k, denoted C_k, forbids cycles with chords as well.

A graph is *four-critical* if it has chromatic number four and every proper subgraph is three-colorable. Ore composition is the edge deletion and vertex splitting operation defined in Section 2. A graph obtained from copies of K_4 by Ore compositions is called a *four-Ore graph* [13, Section 1]. A rotation system specifies the cyclic order of the neighbors at each vertex. For a graph G, write $\chi(G)$ for its chromatic number and $G - e$ for the graph obtained by deleting edge e.

Steinberg conjectured in 1976 that every planar graph without C_4 or C_5 is properly three-colorable; see the account in [4, opening paragraph].

Theorem 1. *There exists an 82-vertex planar graph with 136 edges, without cycles of length four or nine, that is four-critical and is not properly three-colorable. It is a four-Ore graph and has the fewest edges among all 82-vertex four-critical graphs.*

The theorem gives a negative answer to the $i = 9$ case of Kang, Jin and Wang's Problem 1.2 [2], which asks which $i \in \{5, 6, 7, 8, 9\}$ make every planar graph without C_4 or C_i properly three-colorable. Kang, Jin and Wang write that a more general problem "was formulated also in [7]" [2, Section 1, immediately before Problem 1.2]; their reference [7] is Lu, Wang, Wang, Bu, Montassier and Raspaud [1]. Cohen-Addad, Hebdige, Král', Li and Salgado's counterexample to Steinberg's conjecture concerns C_4, C_5 [4, Theorem 3]; its Figure 1 contains the nine-cycle described below. Our

graph answers the C_4, C_9 question using Ore composition and attains Kostochka and Yancey's edge bound [12, Theorem 3]. The contribution is the explicit graph for the length-nine case; the Ore-composition argument and the critical-graph edge bound are the tools. The difficult construction step was to choose identifications between small planar pieces that make their possible proper three-colorings incompatible without creating the forbidden cycles. The restrictions of a piece's three-colorings to the vertices to be identified form a finite list of color patterns. Composing these lists determines which identifications are incompatible with three-colorability; the cycle restrictions must also hold for the resulting graph. The identifications were found by a computer search: small planar pieces were classified by these color patterns, and their compositions were continued until no coloring pattern remained. For the resulting graph, Ore composition gives the short proof of non-three-colorability below; the rotation and exhaustive cycle enumeration establish planarity and the cycle exclusions. Section 2 illustrates the first composition and gives the proof; the complete construction is printed in Appendix G.

Related coloring results. Nearby results on proper, list and defective coloring put the forbidden-pair question in context.

We observe that the G_1 component in Figure 1 of Cohen-Addad and coauthors [4] contains the nine-cycle $a, f, k, n, m, j, i, h, g, a$ in the paper's own labels. This component is retained in the published construction [4, Lemma 2 and proof of Theorem 3].

Here DP-coloring means correspondence coloring [5, Section 1].

Kang, Jin and Wang prove proper three-colorability when C_6 is also excluded [2, Theorem 1.4]. The $\{C_4, C_7, C_9\}$-free class is properly three-colorable by Lu and coauthors, as also recorded in [2, Section 1, paragraph following Problem 1.3]; Kang, Jin and Zhu prove DP-3-colorability of that class [5, Theorem 1.1]. Their Corollary 1.2 states that these graphs are "3-choosable (and hence 3-colorable)" [5].

Defective colorings are a different notion: a (d_1, d_2, d_3)-coloring permits each color class to induce maximum degree at most d_j. Related work includes Jin, Kang, Liu and Wang on $(1, 0, 0)$-colorability [3, Theorem 1.4]. For the four-and-nine pair, Dai, Wang and Xu's title is "Every planar graph without cycles of length 4 or 9 is $(1, 1, 0)$-colorable" [6], while Chen, Dai, Nie, Wang and Yu study $(2, 0, 0)$-colorability [7, Theorem 1.4]. Xiao and Huang's article is entitled "A weak DP-coloring of planar graphs without 4- and 9-cycles" [8]. Jin, Kang, Liu and Wang described the cases $i \in \{6, 7, 8, 9\}$ as "still unsettled" [3, paragraph following Problem 1.2]. For further bibliographic context see Ang and Leo [9], Bu, Wang and Wang [10], and Li, Niu and Tian [11].

The proof depends on finite computations described in Section 3, together with the written arguments below. The proofs, the Lean formalization, the search code, the certificate checkers and the text were produced by AI models (from OpenAI and Anthropic), working under the author's direction. This paper is computer-assisted and has not yet been reviewed by experts.

2 Construction and proof

The graph G has vertex set $\{0, \ldots, 81\}$. We prove non-three-colorability by constructing G from copies of K_4 by Ore composition. A rotation system proves planarity, and enumeration excludes the forbidden cycles. Proper three-colorings of $G - e$ for every edge e establish four-criticality; the number of edges then gives equality in the cited lower bound.

For a graph B, $N_B(z)$ denotes the set of neighbors of vertex z. We use the name Ore composition for the operation called DHGO-composition in Kostochka and Yancey's Section 1 [13].

Take vertex-disjoint graphs A and B, an edge xy of A, and a vertex z of B. Partition $N_B(z)$ into two nonempty disjoint sets N_x, N_y. Delete xy from A, replace z in B by two nonadjacent vertices, and identify those vertices with x, y, respectively, assigning neighbors N_x to x and N_y to y. The resulting graph is an *Ore composition*, also called a Dirac–Hajós–Gallai–Ore (DHGO) composition [13, Section 1].

The key point is that a three-coloring of an Ore composition of two non-three-colorable graphs would force the ends of the deleted edge to have the same color: different colors would give a three-coloring of the first graph. Their common color could then be assigned to the split vertex of the second graph, giving another contradiction. Lemma 1 makes this argument precise, so induction from K_4 proves non-three-colorability. The cycle restrictions are established separately for the resulting graph by the two complete enumerations in Appendix B.

Appendix G represents the construction as a rooted tree of compositions. Tree-node numbers index construction instructions; vertex labels name the vertices in their operands. These are distinct columns of the construction table. Labels at least 82 are temporary split vertices; none survives at the root, node 52. The first composition in the construction uses copies of K_4 on $\{12, 60, 62, 84\}$ and $\{13, 61, 63, 85\}$. It deletes the edge 62 84, splits vertex 85, and assigns neighbor 63 to 62 and neighbors 13, 61 to 84. Figure 1 shows this operation. This is the first internal row of the tree in Appendix G; its later rows specify every subsequent identification. Write J_t for the graph specified at node t of that table. The last composition deletes edge 0 7 from J_{28} and splits vertex 82 of J_{51}: neighbors $2, 3, 4, 5$ are attached to 0 and neighbors $8, 9, 10, 11$ to 7. Its two operands are obtained from J_4, J_{27} and J_{47}, J_{50}, respectively, as shown in Figure 2. These are construction indices, distinct from vertex labels. Figure 3 in Appendix A shows the full graph.

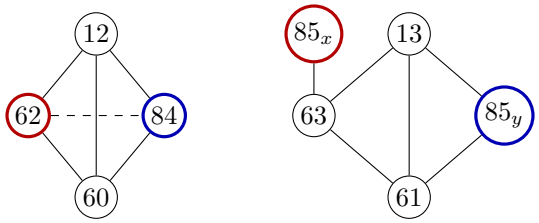


Figure 1: The first composition. Left: edge 62 84 is deleted (dashed). Right: vertex 85 is split. Matching colors indicate the identifications $85_x = 62$ and $85_y = 84$.

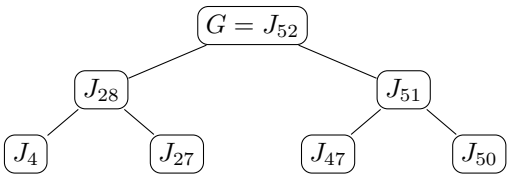


Figure 2: The upper part of the Ore composition tree. At each fork the left graph supplies the deleted edge and the right graph the split vertex. The complete table in Appendix G specifies all 26 compositions.

The operation is the classical construction discussed by Kostochka and Yancey [13, Section 1]; the following argument gives its non-colorability property in the form used here.

Lemma 1. *If neither A nor B is three-colorable, their Ore composition is not three-colorable.*

Proof. Suppose the composite has a proper three-coloring f. Restrict f to $V(A)$. If $f(x) \neq f(y)$, restoring the deleted edge xy gives a proper three-coloring of A, a contradiction. Hence $f(x) = f(y)$. Restrict f to $V(B) \setminus \{z\}$ and give z this common color. Every neighbor of z belonged to N_x or N_y, and therefore has a color different from that common color. Edges not incident with z are unchanged. This is a proper three-coloring of B, again a contradiction. □

The graph K_4 is not three-colorable: four pairwise adjacent vertices need four distinct colors. Inducting over this explicit tree with Lemma 1 proves that G is not three-colorable. Starting with 27 copies of K_4, each join reduces the sum of orders and the sum of sizes by one, so the root has $27 \cdot 4 - 26 = 82$ vertices and $27 \cdot 6 - 26 = 136$ edges. Appendix C verifies that the root is exactly G; Appendices F and G give its edges, rotation and complete construction tree.

2.1 Planarity

The rotation system yields planarity by an orientable-surface argument. Let $D = \{(u, v) : uv \in E(G)\}$ be the darts. Write $\rho_v(u)$ for the neighbor following u in the cyclic order at v. Let F be the set of face-permutation orbits. The face permutation is

$$\phi(u, v) = (v, \rho_v(u)).$$

The face permutation has 56 orbits. To justify this test, place an oriented disk at every vertex, attach an untwisted band for each edge according to the cyclic orders, and cap each boundary walk with a disk. The result is a connected closed orientable surface with a cellular embedding of G. The boundary walks are precisely the orbits of ϕ, so its Euler characteristic is

$$|V| - |E| + |F| = 82 - 136 + 56 = 2.$$

By the classification of closed orientable surfaces this surface is the sphere. Deleting a point outside the embedded graph gives a plane embedding. This proves planarity.

2.2 Cycle exclusion

Two independent complete enumerations of simple cycles give the counts in Appendix B. Both find no cycle of length four or nine. The first enumeration grows every simple path from its smallest vertex s, using only other vertices greater than s. It records a closure to s when the path has at least three vertices and its second vertex is smaller than its last. Every undirected simple cycle has a unique smallest vertex and exactly two orientations, exactly one of which meets this last test. Thus each cycle is counted once. Appendix B gives the second enumeration and the counts.

2.3 Criticality and the edge bound

The deletion colorings establish criticality, and the cited lower bound then gives the stated edge minimum.

Proof of Theorem 1. Induction from K_4 using Lemma 1 proves that G is not three-colorable and is a four-Ore graph. The rotation system has 56 faces, so $82 - 136 + 56 = 2$; the orientable-surface argument in Section 2.1 proves planarity. The enumerations in Section 2.2 and Appendix B find no simple four- or nine-cycle. For an edge e, use the coloring of $G - e$ specified in Appendix C. The two endpoints of e necessarily have equal colors, since otherwise this would color G. Recoloring one endpoint with a fourth color yields a proper four-coloring of G. Together with the preceding induction, this proves $\chi(G) = 4$. The graph has no isolated vertices. Any proper subgraph omits some edge of G, even if it omits a vertex, and is therefore a subgraph of some $G - e$. Restricting that deletion coloring proves its three-colorability. Consequently G is four-critical. Finally Kostochka and Yancey's bound gives $|E(H)| \geq \lceil (5|V(H)| - 2)/3 \rceil$ for every four-critical graph H, and $136 = (5 \cdot 82 - 2)/3$ proves the edge minimum at order 82. □

3 Computational verification

The computations used above establish the face orbits of the rotation, the cycle counts in Appendix B, the compositions in Appendix G, and a proper three-coloring of $G - e$ for every edge e. Appendix D gives two further non-colorability arguments: exact elimination of coloring constraints and a clausal proof of unsatisfiability.

The formal graph G has vertex set $\{0, \ldots, 81\}$ and adjacency given by exactly the 136 unordered edges in Appendix F. The Lean theorem `Steinberg.planar_counterexample` states the following conjunction for this graph:

$$G \text{ has a crossing-free straight-line drawing with rational coordinates;}$$
$$\text{every simple cycle of } G \text{ has length different from four and nine;}$$
$$\text{there is no proper coloring } V(G) \longrightarrow \{0, 1, 2\}.$$

The formal planarity predicate, `RationalPlanar`, requires an injective assignment of rational coordinate pairs to the vertices, with a vertex on an edge segment only at an endpoint and edge segments meeting only at a shared endpoint. Segments are closed real affine segments between those rational points. The proof checks the existing drawing's six-decimal coordinates as exact rationals. Mathlib's simple closed walks and proper colorings supply the other two predicates; their conventions agree with the introduction, including cycles with chords. This proves the planar counterexample to the length-nine question directly from these definitions.

The formal theorem has no hypotheses and is checked by Lean's kernel using only its standard axioms. The exact axiom inventory, pinned versions and build commands are in the accompanying checking notes. Appendix E describes the formal non-colorability argument. The remaining assertions of Theorem 1—Ore membership, criticality and the edge minimum—use the written arguments in Section 2; the last uses Kostochka and Yancey's theorem. The Ore construction, deletion colorings and other cycle counts also have independent executable checks.

Appendix A The embedding certificate

Appendix F gives the edge list and cyclic neighbor orders. Tracing the face permutation of Section 2.1 gives 56 orbits, after checking simplicity, the neighbor orders and connectedness.

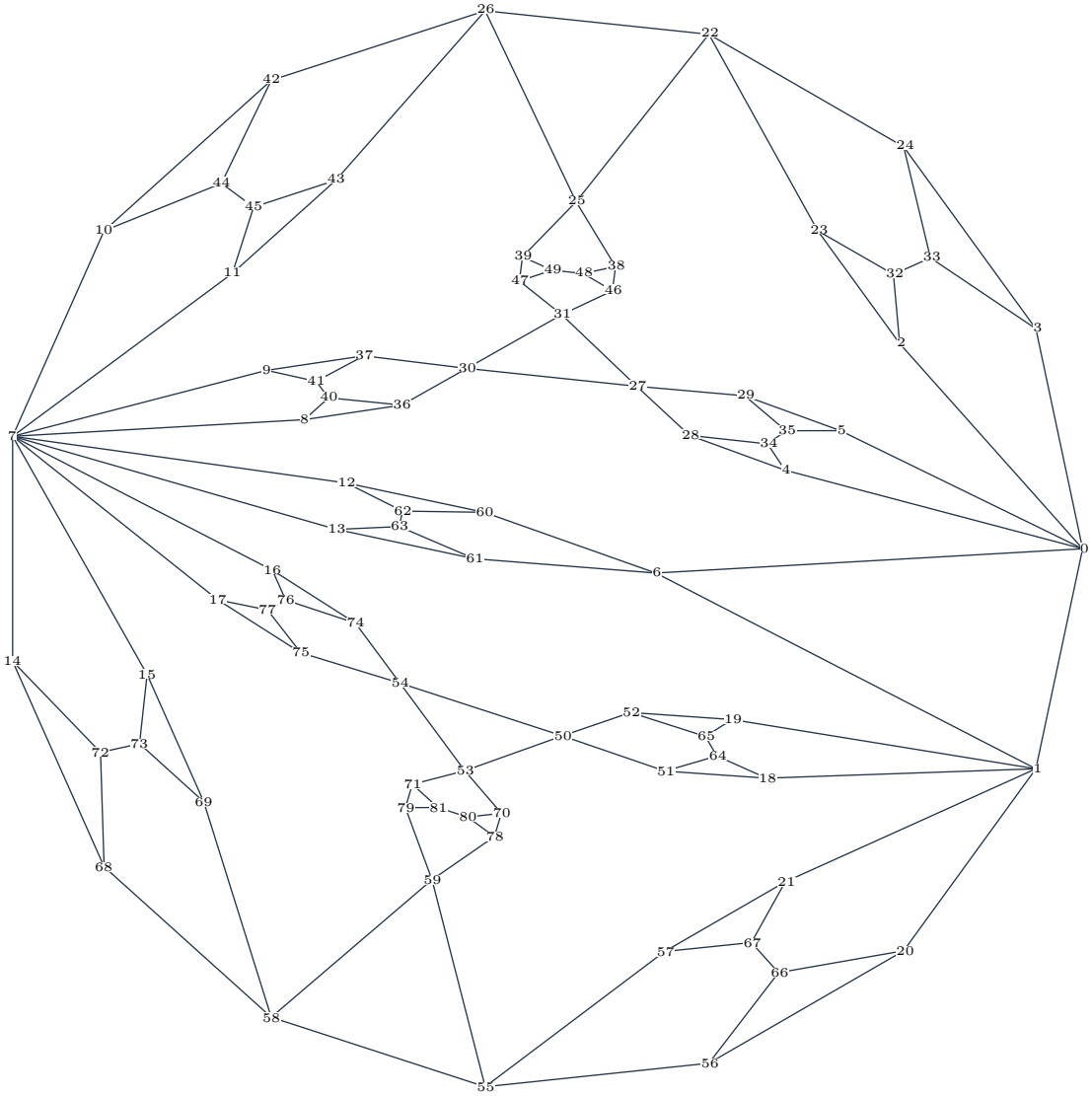


Figure 3: The counterexample, with vertex numbers from the accompanying SVG drawing. The drawing was generated from, and checked against, the edge and rotation data in Appendix F; its rational coordinates are also checked in Lean.

Drawing. The figure is obtained by applying Tutte's barycentric method to a triangulation of the bounded faces, using temporary centers, and then omitting the added vertices and edges. For background see Tutte's "How to Draw a Graph" [17]. The finite-decimal coordinates are exact rationals, checked in Lean as described in Section 3.

Appendix B Excluding cycles

An independent enumeration checks the cycle counts used in Section 2.2, counting undirected simple cycles without multiplicity. The second enumeration instead chooses an edge (s, t) with $s < t$, and enumerates simple s–t paths avoiding that edge, with all other vertices greater than s. Closing by ts gives a simple cycle. A cycle whose smallest vertex is s is found twice, once for each of its two edges at s; division by two gives the count. Each raw count is even. Both enumerations run through length 14 and return the same table:

length	3	4	5	6	7	8	9	10	11	12	13	14
count	27	0	22	55	44	11	0	0	64	320	832	1824

In particular there is no C_4, C_9 or C_{10}.

Appendix C The construction and deletion colorings

The construction table identifies the graph used in the Ore induction, and the deletion colorings certify its criticality. The tree printed in Appendix G has 27 leaves and 26 internal nodes. Each leaf gives four distinct vertices and all six edges of a K_4. Every internal node gives two earlier, disjoint operands, an edge xy of the first, a vertex z of the second, and two nonempty sets N_x, N_y partitioning $N_B(z)$. Applying the operation in Section 2 at each node gives the root graph. Every non-root node occurs as an operand exactly once. At the root the vertex labels are $0, \dots, 81$ and the edges are exactly those in Appendix F, with the identity correspondence on vertices.

For each of the 136 edges e, the accompanying deletion-coloring data gives a vector $(c_0, \dots, c_{81}) \in \{0, 1, 2\}^{82}$ whose coordinates differ across every edge other than e. These are the deletion colorings used in the proof of Theorem 1; the checking notes specify their location and verification.

C.1 Edge count and other cycle lengths

The tree proves that G is a four-Ore graph. The bound used in the main proof is Kostochka and Yancey's Theorem 3 [12], also stated as Theorem 4 in [13]. Every four-critical graph H satisfies

$$|E(H)| \geq \left\lceil \frac{5|V(H)| - 2}{3} \right\rceil.$$

For the graph in this paper,

$$136 = \frac{5 \cdot 82 - 2}{3}.$$

The general four-Ore construction and its edge count are discussed in Kostochka and Yancey's Section 1 [13]. Thus the edge minimum in Theorem 1 is at fixed order 82, among all four-critical graphs. The cycle table also gives the exclusion of C_{10} and the presence of C_5, C_6, C_7, C_8.

Appendix D Two additional checks of non-colorability

These checks establish non-colorability independently of the Ore induction.

D.1 Exact constraint elimination

A relation's *scope* is the set of vertices whose colors it specifies. The *interaction graph* joins two remaining vertices when they occur in the scope of a common relation.

For each edge uv start with the relation $R_{uv} = \{(a, b) \in \{0, 1, 2\}^2 : a \neq b\}$. Maintain a conjunction of relations on the remaining vertices. To eliminate v, collect all relations mentioning it, let S be the union of their scopes minus v, and replace them by the relation

$$R_S = \{a \in \{0, 1, 2\}^S : \exists c \in \{0, 1, 2\} \bigwedge_{R \text{ mentioning } v} R(a, v = c)\}.$$

All other relations remain. The new conjunction is exactly the existential projection of the old one. Induction on eliminations shows that satisfiability is preserved. An empty relation proves that no coloring exists; a nonempty relation on the empty scope is a tautology and may be discarded.

Each elimination enumerates all $3^{|S|}$ assignments and all three values of c. A vertex of minimum degree in the current interaction graph is chosen, breaking ties by vertex label. This choice affects cost, not correctness. For this graph an empty relation is reached; the largest remaining scope S has size three.

D.2 Exact graph-to-CNF reduction and symmetry

The coloring constraints form a conjunction of clauses, or conjunctive normal form (CNF). Independently introduce Boolean variables $x_{v,c}$, numbered $3v + c + 1$, for $0 \leq v < 82$ and $c \in \{0, 1, 2\}$. The generator adds the clauses

$x_{v,0} \vee x_{v,1} \vee x_{v,2}$	for each v,
$\neg x_{v,a} \vee \neg x_{v,b}$	for each v and $0 \leq a < b < 3$,
$\neg x_{u,c} \vee \neg x_{v,c}$	for each $uv \in E(G)$ and c.

The first two lines specify exactly one color at each vertex, and the third specifies distinct colors across every edge. Assigning the unique true color is therefore a bijection between satisfying assignments of these clauses and proper three-colorings of G.

The lexicographically first triangle is $(0, 1, 6)$. The generator also adds $x_{0,0}$, $x_{1,1}$ and $x_{6,2}$, that is, unit clauses 1, 5 and 21. Every proper coloring gives distinct colors to this triangle. There is a permutation of the three color names taking them, in this order, to $0, 1, 2$; the other clauses are invariant under this permutation. Adding these units thus preserves satisfiability.

The resulting formula has a clausal proof ending with the empty clause, generated with CaDiCaL [15] and checked by DRAT-trim [14]. Each clause addition satisfies a checked redundancy condition preserving satisfiability; a clause deletion weakens the formula. The final empty clause therefore proves unsatisfiability. Together with the color-permutation argument and the exact reduction above, this proves that G has no proper three-coloring. The formula and proof, their sizes and the checking commands are given in the checking notes.

Appendix E Formal verification

The formal proof certifies non-three-colorability by deducing a contradiction from the coloring clauses. Its coloring formula has the edge and vertex clauses of Appendix D, without units that prescribe a color for an individual vertex. The formal reduction proves that a proper coloring would satisfy this formula. Equality of the formula produced by the reduction with the certificate's input is proved in Lean.

Unit propagation assigns the only remaining literal of a clause when all its other literals are false. Clause additions are justified by reverse unit propagation: assuming the negation of a proposed clause, unit propagation in previously justified clauses yields a contradiction. The formal proof establishes this implication and applies it to every addition. Hints specify the clauses used in these deductions. Deletion instructions leave earlier clauses available; every such clause remains entailed by the original input. The final empty clause proves unsatisfiability of that input.

The use of hints follows Cruz-Filipe, Heule, Hunt, Kaufmann and Schneider-Kamp [16, abstract], whose format extends DRAT, a clausal proof format allowing clause additions and deletions, "with hints that facilitate a simple and fast validation algorithm." The checking notes give the exact certificate formats and formal theorem names.

Appendix F Explicit graph and rotation

The graph list gives every edge, and the rotation gives every neighbor in cyclic order. Together they specify the labeled graph and its embedding.

Graph

```
82 136
0 1
0 2
0 3
0 4
0 5
0 6
1 6
1 18
1 19
1 20
1 21
2 23
2 32
3 24
3 33
4 28
4 34
5 29
5 35
6 60
6 61
7 8
7 9
7 10
7 11
7 12
7 13
7 14
7 15
7 16
7 17
8 36
8 40
9 37
9 41
10 42
10 44
11 43
11 45
12 60
12 62
13 61
13 63
14 68
14 72
15 69
15 73
16 74
16 76
17 75
17 77
18 51
18 64
19 52
19 65
20 56
20 66
21 57
21 67
22 23
22 24
22 25
22 26
23 32
24 33
25 26
25 38
25 39
26 42
26 43
27 28
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27 30
27 31
28 34
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54 74
54 75
55 56
55 57
55 58
55 59
56 66
57 67
58 59
58 68
58 69
59 78
59 79
60 62
61 63
62 63
64 65
66 67
68 72
69 73
70 78
70 80
71 79
71 81
72 73
74 76
75 77
76 77
78 80
79 81
80 81
```

Rotation

```
0: 1 3 2 5 4 6
1: 0 6 19 18 21 20
2: 23 0 32
3: 33 0 24
4: 28 0 34
5: 35 0 29
6: 1 0 60 61
7: 12 8 9 11 10 14 15 17 16 13
8: 7 36 40
9: 37 7 41
10: 42 7 44
11: 45 7 43
12: 60 7 62
13: 7 61 63
14: 7 68 72
15: 69 7 73
16: 74 7 76
17: 77 7 75
18: 51 1 64
19: 65 1 52
20: 56 1 66
21: 67 1 57
22: 25 23 24 26
23: 22 2 32
24: 3 22 33
25: 38 22 26 39
26: 22 42 43 25
27: 30 28 29 31
28: 27 4 34
29: 5 27 35
30: 36 27 31 37
31: 27 46 47 30
32: 2 33 23
33: 32 3 24
34: 4 35 28
35: 34 5 29
36: 8 30 40
37: 30 9 41
38: 46 25 48
39: 25 47 49
40: 41 8 36
41: 9 40 37
42: 26 10 44
43: 11 26 45
44: 10 45 42
45: 44 11 43
46: 31 38 48
47: 39 31 49
48: 49 46 38
49: 47 48 39
50: 53 51 52 54
51: 50 18 64
52: 19 50 65
53: 70 50 54 71
54: 50 74 75 53
55: 58 56 57 59
56: 55 20 66
57: 21 55 67
58: 68 55 59 69
59: 55 78 79 58
60: 6 12 62
61: 13 6 63
62: 63 60 12
63: 61 62 13
64: 18 65 51
65: 64 19 52
66: 20 67 56
67: 66 21 57
68: 14 58 72
69: 58 15 73
70: 78 53 80
71: 53 79 81
72: 73 14 68
73: 15 72 69
74: 54 16 76
75: 17 54 77
76: 16 77 74
77: 76 17 75
78: 59 70 80
79: 71 59 81
80: 81 78 70
81: 79 80 71
```

Appendix G The explicit Ore tree

Each leaf row specifies the four vertices of a complete graph. Each internal row specifies the two operand indices, then x, y, z, N_x, N_y in the notation of Section 2. Operands are disjoint at every composition. The root graph is G, with the identity correspondence on $0, \ldots, 81$.

node	vertices of K_4
0	$0, 1, 6, 7$
1	$12, 60, 62, 84$
2	$13, 61, 63, 85$
5	$50, 53, 54, 83$
6	$16, 74, 76, 89$
7	$17, 75, 77, 90$
10	$55, 58, 59, 88$
11	$14, 68, 72, 92$
12	$15, 69, 73, 93$
15	$20, 56, 66, 91$
16	$21, 57, 67, 94$
20	$70, 78, 80, 87$
21	$71, 79, 81, 95$
24	$18, 51, 64, 86$
25	$19, 52, 65, 96$
29	$22, 25, 26, 82$
30	$10, 42, 44, 100$
31	$11, 43, 45, 101$
34	$27, 30, 31, 99$
35	$8, 36, 40, 103$
36	$9, 37, 41, 104$
39	$4, 28, 34, 102$
40	$5, 29, 35, 105$
44	$38, 46, 48, 98$
45	$39, 47, 49, 106$
48	$2, 23, 32, 97$
49	$3, 24, 33, 107$

node	A	B	x	y	z	N_x	N_y
3	1	2	62	84	85	63	13, 61
4	0	3	6	7	84	60, 61	12, 13
8	6	7	76	89	90	77	17, 75
9	5	8	54	83	89	74, 75	16, 17
13	11	12	72	92	93	73	15, 69
14	10	13	58	88	92	68, 69	14, 15
17	15	16	66	91	94	67	21, 57
18	14	17	55	88	91	56, 57	20, 21
19	9	18	53	83	88	59	14, 15, 20, 21
22	20	21	80	87	95	81	71, 79
23	19	22	53	59	87	70, 71	78, 79
26	24	25	64	86	96	65	19, 52
27	23	26	50	83	86	51, 52	18, 19
28	4	27	1	7	83	18, 19, 20, 21	14, 15, 16, 17
32	30	31	44	100	101	45	11, 43
33	29	32	26	82	100	42, 43	10, 11
37	35	36	40	103	104	41	9, 37
38	34	37	30	99	103	36, 37	8, 9
41	39	40	34	102	105	35	5, 29
42	38	41	27	99	102	28, 29	4, 5
43	33	42	25	82	99	31	4, 5, 8, 9
46	44	45	48	98	106	49	39, 47
47	43	46	25	31	98	38, 39	46, 47
50	48	49	32	97	107	33	3, 24
51	47	50	22	82	97	23, 24	2, 3
52	28	51	0	7	82	2, 3, 4, 5	8, 9, 10, 11

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Kostochka and Yancey describe the construction as “found in Dirac’s paper” and “used by Gallai . . . and Hajós”, and say that Ore “used it for a composition of complete graphs” [13, Section 1]. The sharp edge bound is Kostochka and Yancey’s [12, Theorem 3], also reproduced as Theorem 4 in [13]. We acknowledge the researchers cited above for posing and developing the forbidden-cycle questions, and Cohen-Addad and coauthors for their disproof of Steinberg’s conjecture. The formalization uses Lean 4 and its standard library and Mathlib, developed by their contributors. The SAT certificate was generated with CaDiCaL [15] and checked with Heule and Wetzler’s DRAT-trim [14].

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