A counterexample to Stanley's conjecture on Smith forms of Jucys–Murphy elements

P. Graham
https://orcid.org/0009-0007-8493-1505

October 24, 2026

Abstract

We disprove Stanley's conjectured Smith-form formula for Jucys–Murphy elements on integral Specht modules: for shape $(3,1,1)$ and $k=3$, the Smith form is $(1,1,1,1,2,2)$ instead of $(1,1,1,1,1,4)$. A second example, of shape $(5,2)$, disproves his conjectural remark that every invariant factor is a product of some of the eigenvalues. For $n \geq 3$, we prove in writing that the formula holds at $k=3$ precisely for partitions with at most two rows or at most two columns. The classification holds on every full-rank lattice in the rational Specht module stable under the permutations of $\{1,2,3\}$. Stanley's formula was based on Darij Grinberg's computations. Grinberg also clarified the integral lattice, and Alexander Chervov asked about the conjecture's status. The elements first appeared in Alfred Young's work and were rediscovered by Jucys and Murphy; Vershik and Okounkov give their common eigenbasis and tableau-content eigenvalues. The proof combines those eigenvalues with an integral group-ring identity and explicit integer matrices. Lean verifies the finite matrix calculations and supporting algebraic identities; independent programs identify the matrices with the Specht-module actions outside Lean.

1 Introduction

Write $\lambda \vdash n$ for a partition $\lambda = (\lambda_1, \lambda_2, \dots)$ of n, with $\lambda_1 \geq \lambda_2 \geq \cdots$. A standard Young tableau fills its diagram with $1, \dots, n$, increasing along rows and columns; their number is f^λ. The cell (i,j) is in row i and column j, and $j-i$ is its content. The symmetric group on $\{1,\dots,n\}$ is denoted by $\mathfrak{S}_n$, and $\mathbb{Z}\mathfrak{S}_n$ is its integral group ring, consisting of integer linear combinations of permutations.

A row tabloid identifies tableaux whose entries differ only in their order within each row. Let $M^\lambda_\mathbb{Z}$ be the free $\mathbb{Z}$-module on the row tabloids $\{T\}$ of shape λ, with $\mathfrak{S}_n$ acting by $\pi\{T\} = \{\pi T\}$, where πT replaces each entry x of T by $\pi(x)$. The column stabilizer consists of the permutations preserving each column as a set. For a tableau T with column stabilizer C_T, the polytabloid is $e_T = \sum_{\pi \in C_T} \operatorname{sgn}(\pi)\, \pi\{T\}$, and the integral Specht module $S^\lambda_\mathbb{Z}$ is the $\mathbb{Z}$-span of all e_T. The standard polytabloids e_T, T standard, form a $\mathbb{Z}$-basis of $S^\lambda_\mathbb{Z}$ ([4, Theorem 8.4 and Corollary 8.6], [3, Theorem 5.9.1]); Sagan's complex-field version is [8, Theorem 2.6.5(1)]. Its rational extension is $S^\lambda_\mathbb{Q}$. For $1 \leq k \leq n$, the Jucys–Murphy element is $X_k = (1,k) + \cdots + (k-1,k)$, with $X_1 = 0$; its action is the sum of these permutation actions.

An integer matrix is unimodular when its determinant is 1 or -1. The Smith normal form is a diagonal matrix obtained by unimodular row and column operations. Its diagonal entries, the invariant factors, are denoted by $\alpha_1, \dots, \alpha_{f^\lambda}$. Smith forms are written with nonnegative entries in divisibility order, $\alpha_1 \mid \alpha_2 \mid \cdots$, so that any zeros come last; an empty product is 1.

The matrix $W_{\lambda,k}$ is that of X_k in the standard polytabloid basis; SYT means standard Young tableau and $\overset{\text{snf}}{\to}$ denotes reduction to Smith normal form. In his survey *Smith normal form in combinatorics*, Stanley states Conjecture 7.4 as follows [9, Conjecture 7.4]:

> Let $\lambda \vdash n$, $1 \le k \le n$, and $W_{\lambda,k} \overset{\text{snf}}{\to} (\alpha_1, \dots, \alpha_{f^\lambda})$. Fix $1 \le r \le f^\lambda$. Let S_r be the set of positions (i,j) that k occupies in at least r of the SYT's of shape λ. Then $\alpha_{f^\lambda - r+1} = \pm \prod_{(i,j)\in S_r}(j-i)$.

Theorem 1 (Smith form checked in Lean; Specht-module matrix checked by two certificate checkers)**.** *For $\lambda = (3,1,1)$ and $k = 3$, the Smith normal form of X_3 acting on the integral Specht module $S_{\mathbb{Z}}^{(3,1,1)}$ is $(1,1,1,1,2,2)$. Conjecture 7.4 predicts $(1,1,1,1,1,4)$. Hence Conjecture 7.4 is false.*

A rational eigenvalue description does not determine the integral Smith form. The conjecture and its worked example are Stanley's [9, Conjecture 7.4 and following example]. He credits Darij Grinberg's computations for $\lambda \vdash n \le 7$ in the paragraph preceding the conjecture. Alexander Chervov's Question 1 drew attention to their status by asking about progress on Stanley's conjectures [1, Question 1]. Darij Grinberg clarified the intended lattice as the integral Specht module in the standard tableau basis [1, comment 1262855]; the lattice used here follows his clarification. The remark following Conjecture 7.4 [9, Section 7, page 15, remark following Conjecture 7.4] conjectures that every invariant factor is a product of some of the eigenvalues. The examples below lie within the reported range $n \le 7$; the underlying unpublished data remain unavailable for comparison. We attribute no error to them.

Stanley [9, Section 7, paragraph preceding Conjecture 7.4] traces the elements to Alfred Young. The independent rediscovery by Jucys and Murphy [5, 7] is reported by Vershik and Okounkov [10, Introduction, page 4]. We use Vershik and Okounkov's common eigenbasis indexed by standard tableaux [10, Corollary 2.10 and the following paragraph]. In that basis, X_k has eigenvalue the content of the cell occupied by k [10, Theorem 5.8]. This common rational eigenbasis is Young's seminormal basis [10, Proposition 6.1 and following paragraph]; it is related to the standard polytabloid basis by a change of basis over $\mathbb{Q}$. On Grinberg's interpretation of $W_{\lambda,k}$ as the Specht-module matrix, Stanley's content-spectrum statement gives these same eigenvalues [9, Section 7, page 15, content-spectrum paragraph preceding Conjecture 7.4].

By a lattice in $S_{\mathbb{Q}}^\lambda$ we mean a full-rank $\mathbb{Z}$-lattice, that is, a finitely generated $\mathbb{Z}$-submodule spanning $S_{\mathbb{Q}}^\lambda$ over $\mathbb{Q}$, equivalently a free $\mathbb{Z}$-module of rank f^λ spanning $S_{\mathbb{Q}}^\lambda$; the Smith form of an endomorphism of a lattice is that of its matrix in any $\mathbb{Z}$-basis. A lattice is stable under a set of permutations when each permutation maps it into itself.

The main difficulty is controlling the integral lattice: a rational eigenbasis need not be a unimodular integral basis. The new step is to use $X_3(X_3 - (1,2)) = 2$ in the integral group ring. It forces every invariant factor to divide 2 on each lattice stable under these three letters. Counting the tableaux where 3 has content 2 or -2 determines the determinant and hence the number of factors 2. The conjecture gives a factor 4 precisely when both cells $(1,3)$ and $(3,1)$ occur, yielding the classification. For shape $(5,2)$, the eigenvalues are 0 and 4, while unimodular transformations of the integral matrix produce an invariant factor 2; this disproves the conjectural remark as well. The construction and computation are given below, followed by the classification proof.

The proof uses explicit computations and a partial Lean formalization, whose statements and relation to the written argument are described in Section 5. The proofs, the Lean formalization, the certificate checkers and the text were produced by AI models (from OpenAI and Anthropic), working under the author's direction. The paper has a written proof, with Lean and computer checks; it has not yet been reviewed by experts.

2 The matrix of X_3

We construct the matrix in Theorem 1 in the standard polytabloid basis.

Let $\lambda = (3,1,1)$, so $n = 5$, $f^\lambda = 6$ and $X_3 = (1,3) + (2,3)$. Order the standard tableaux as in Figure 1.

$$T_1 = \begin{array}{|c|c|c|}\hline 1&2&3\\\hline 4\\\cline{1-1} 5\\\cline{1-1}\end{array}, \quad T_2 = \begin{array}{|c|c|c|}\hline 1&2&4\\\hline 3\\\cline{1-1} 5\\\cline{1-1}\end{array}, \quad T_3 = \begin{array}{|c|c|c|}\hline 1&2&5\\\hline 3\\\cline{1-1} 4\\\cline{1-1}\end{array}, \quad T_4 = \begin{array}{|c|c|c|}\hline 1&3&4\\\hline 2\\\cline{1-1} 5\\\cline{1-1}\end{array}, \quad T_5 = \begin{array}{|c|c|c|}\hline 1&3&5\\\hline 2\\\cline{1-1} 4\\\cline{1-1}\end{array}, \quad T_6 = \begin{array}{|c|c|c|}\hline 1&4&5\\\hline 2\\\cline{1-1} 3\\\cline{1-1}\end{array}$$

Figure 1: The standard tableaux of shape $(3,1,1)$ in the order used for A.

For each T_j, expand e_{T_j} as the signed sum over its column stabilizer, apply X_3 to the tabloids, and express the result in the standard polytabloid basis. Let column j of A hold the coordinates of $X_3 e_{T_j}$ in the standard polytabloid basis, so that $X_3 e_{T_j} = \sum_i A_{ij} e_{T_i}$.

$$A = \begin{pmatrix} 2 & 0 & 0 & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & -1 & -2 \end{pmatrix}.$$

For example, the first column of A says $X_3 e_{T_1} = 2e_{T_1} - e_{T_2} + e_{T_3}$.

The conjectured Smith form. The entry 3 occupies the cell $(1,3)$ in T_1, $(2,1)$ in T_2 and T_3, $(1,2)$ in T_4 and T_5, and $(3,1)$ in T_6. Hence $S_1 = \{(1,2),(1,3),(2,1),(3,1)\}$ with $|\prod(j-i)| = |1 \cdot 2 \cdot (-1) \cdot (-2)| = 4$, $S_2 = \{(1,2),(2,1)\}$ with product of absolute value 1, and $S_r = \emptyset$ for $3 \le r \le 6$. Conjecture 7.4 therefore predicts $(1,1,1,1,1,4)$.

A unitriangular matrix is triangular with every diagonal entry equal to 1.

Proof of Theorem 1. The matrix A is lower triangular, so $\det A = 2 \cdot (-1) \cdot (-1) \cdot 1 \cdot 1 \cdot (-2) = -4$. Modulo 2, the first row of A vanishes and the sum of rows 2 to 6 vanishes, while rows 2 to 5 restricted to columns 2 to 5 form a unitriangular matrix; hence A has rank 4 over $\mathbb{F}_2$. So exactly two invariant factors are even, their product with the others is 4, and the form is $(1,1,1,1,2,2)$. The predicted form has only one even factor and would require rank 5 over $\mathbb{F}_2$. □

3 An invariant factor that is not a product of eigenvalues

The second example has an invariant factor that is not a product of eigenvalues.

Proposition 2 (Smith form and eigenvalues checked in Lean; Specht-module matrix checked by two certificate checkers)**.** *Let $\lambda = (5,2)$ and $k = 7$, so $n = 7$ and $f^\lambda = 14$. On $S^{(5,2)}_{\mathbb{Q}}$, $X_7(X_7 - 4) = 0$, and the eigenvalues of X_7 are 0, with multiplicity 5, and 4, with multiplicity 9. The Smith normal form of X_7 on $S^{(5,2)}_{\mathbb{Z}}$ is*

$$(1,1,1,1,2,4,4,4,4,0,0,0,0,0),$$

while Conjecture 7.4 predicts $(1,1,1,1,1,4,4,4,4,0,0,0,0,0)$. The invariant factor 2 is not a product of eigenvalues, so Stanley's weaker remark also fails, for a two-row shape with $k = n$.

Order the standard tableaux by their second rows:

$$(6,7),(5,7),(5,6),(4,7),(4,6),(4,5),(3,7),$$
$$(3,6),(3,5),(3,4),(2,7),(2,6),(2,5),(2,4).$$

The first row is the increasing complement in $\{1,\dots,7\}$. With the same column convention and tabloid expansion as above, this constructs the matrix A_{52} of X_7. Its entries are listed in Appendix A.

Let $\lambda=(5,2)$ and $k=n=7$, so $f^\lambda=14$. In a standard tableau of shape $(5,2)$ the entry 7 is the largest, so it occupies a corner: $(1,5)$, of content 4, in the 9 tableaux obtained from the standard tableaux of shape $(4,2)$, or $(2,2)$, of content 0, in the 5 obtained from those of shape $(5,1)$. So the eigenvalues of X_7 on $S^{(5,2)}_{\mathbb{Q}}$ are 4, with multiplicity 9, and 0, with multiplicity 5, and since X_7 acts diagonally on Young's seminormal basis [10, Theorem 5.8], $X_7(X_7-4)=0$ there. The integer matrix A_{52} constructed below satisfies $A_{52}(A_{52}-4I)=0$; its eigenvalues and Smith form are checked as described in Section 5.

The conjectured Smith form. Using the corner counts above, the sets in Conjecture 7.4 are $S_r=\{(1,5),(2,2)\}$ for $r\le 5$, with product 0; $S_r=\{(1,5)\}$ for $6\le r\le 9$, with product 4; and $S_r=\emptyset$ for $r\ge 10$. Hence the prediction is $\alpha_{14}=\cdots=\alpha_{10}=0$, $\alpha_9=\cdots=\alpha_6=4$ and $\alpha_5=\cdots=\alpha_1=1$, that is, $(1,1,1,1,1,4,4,4,4,0,0,0,0,0)$.

The Smith form. The unimodular matrices U_{52},V_{52} in Appendix A satisfy

$$U_{52}A_{52}V_{52}=\operatorname{diag}(1,1,1,1,2,4,4,4,4,0,0,0,0,0).$$

Left and right multiplication by these unimodular matrices preserves the Smith form. The displayed diagonal has nine nonzero entries, giving rank 9, as the eigenvalues require. Its entries are nonnegative and each divides the next, so it is already the Smith form; its fifth entry gives $\alpha_5=2$. Every product of eigenvalues is 1, a power of 4, or 0, so 2 is not one of them: the weaker remark fails, and so does the conjecture itself. Since every unimodular diagonalization of the matrix has an entry divisible by 2 but not by 4, the failure does not depend on how the diagonal form is normalized.

4 The case $k=3$

Here $v_2(d)$ denotes the exponent of 2 in a nonzero integer d. In exponent formulas, $[\,\cdot\,]$ is 1 if the condition holds and 0 otherwise. The integral group-ring identity below determines the Smith form on each lattice in the theorem. In the displayed Smith form, exponents denote repetition of an entry.

Theorem 3 (proved in writing; not machine-checked)**.** *Let $\lambda\vdash n\ge 3$, and let m be the number of standard Young tableaux of shape λ in which 3 occupies the cell $(1,3)$ or $(3,1)$. On every lattice in $S^\lambda_{\mathbb{Q}}$ that is stable under the permutations of $\{1,2,3\}$, in particular on $S^\lambda_{\mathbb{Z}}$, the Smith normal form of X_3 is $(1^{f^\lambda-m},2^m)$.*

Corollary 4 (proved in writing; not machine-checked)**.** *Let $\lambda\vdash n\ge 3$. Conjecture 7.4 holds at $k=3$ if and only if $\lambda_1\le 2$ or λ has at most two rows. When $\lambda_1\ge 3$ and λ has at least three rows, it fails on every lattice as in Theorem 3: the predicted largest invariant factor is 4, while every actual one divides 2.*

Lemma 5 (checked in Lean)**.** *In* $\mathbb{Z}\mathfrak{S}_n$, $n \geq 3$, *let* $u = (1,2)$. *Then* $X_3(X_3 - u) = (X_3 - u)X_3 = 2$.

Proof. Write $s = (1,3)$ and $t = (2,3)$, so $X_3 = s + t$. Let c and c^2 be the two 3-cycles on $\{1,2,3\}$. Then $X_3^2 = s^2 + t^2 + st + ts = 2 + c + c^2$, since st and $ts = (st)^{-1}$ are the two 3-cycles. Each of su, tu, us, ut is a product of two distinct transpositions in $\mathfrak{S}_3$, hence a 3-cycle, and $su \neq tu$, $us \neq ut$. So $X_3 u = u X_3 = c + c^2$, and $X_3(X_3 - u) = (X_3 - u)X_3 = 2$. □

Proof of Theorem 3 (written). Let L be a lattice in $S^\lambda_{\mathbb{Q}}$ stable under the permutations of $\{1,2,3\}$. Both X_3 and $X_3 - u$ lie in $\mathbb{Z}\mathfrak{S}_3$, so in a $\mathbb{Z}$-basis of L they act by integer matrices A and B, and $AB = BA = 2I$ by Lemma 5. Choose unimodular P, Q with $PAQ = D = \operatorname{diag}(d_1, \ldots, d_f)$, $f = f^\lambda$. Write $C = Q^{-1}BP^{-1}$. Then $D \cdot (Q^{-1}BP^{-1}) = 2I$ with $Q^{-1}BP^{-1}$ an integer matrix, so $d_i C_{ii} = 2$ for an integer C_{ii}, and each d_i is 1 or 2. The number of d_i equal to 2 is the 2-adic valuation of $|\det A|$.

The determinant of X_3 on $S^\lambda_{\mathbb{Q}}$ does not depend on L. By the description of the eigenvalues of the Jucys–Murphy elements given by Vershik and Okounkov [10, Theorem 5.8], X_3 acts diagonally on Young's seminormal basis of $S^\lambda_{\mathbb{Q}}$, with eigenvalue on the vector indexed by a standard tableau T equal to the content of the cell of 3 in T. In a standard tableau the entries $1, 2, 3$ fill a partition of 3, so 3 occupies $(1,2)$, $(2,1)$, $(1,3)$ or $(3,1)$, of contents $1, -1, 2, -2$. Hence $|\det A| = 2^m$, and the Smith form is $(1^{f-m}, 2^m)$. □

Proof of Corollary 4 (written). For a cell c let N_c be the number of standard tableaux of shape λ in which 3 occupies c ($N_c = 0$ if $c \notin \lambda$), so that $m = N_{(1,3)} + N_{(3,1)}$. Every S_r is contained in $\{(1,2),(2,1),(1,3),(3,1)\}$, and the first two cells have contents ± 1. Hence the prediction is

$$\alpha_{f-r+1} = 2^{[N_{(1,3)} \geq r] + [N_{(3,1)} \geq r]} \qquad (1 \leq r \leq f),$$

Suppose $\lambda_1 \geq 3$ and λ has at least three rows. The standard fillings of size three that place $1, 2, 3$ along the first row, or down the first column, put 3 in $(1,3)$, respectively $(3,1)$; both shapes lie in λ, and each filling extends to a standard tableau of shape λ by filling the remaining cells with $4, \ldots, n$ row by row, top to bottom and left to right. So both contents 2 and -2 occur, $N_{(1,3)} \geq 1$ and $N_{(3,1)} \geq 1$, and the prediction gives $\alpha_f = 4$. By Theorem 3 every invariant factor divides 2, so the conjecture fails, on every lattice of the theorem.

Otherwise at most one of $(1,3)$ and $(3,1)$ lies in λ, and the corresponding N_c equals m (if neither does, $m = 0$). The prediction is then $\alpha_{f-r+1} = 2$ for $r \leq m$ and 1 for $r > m$, that is, $(1^{f-m}, 2^m)$, which is the Smith form given by Theorem 3. □

5 Verification and reproducibility

Here we describe the matrix statements verified in Lean [2], using mathlib [6], and their relation to the Specht-module calculations. For the literal integer matrices A, U, V of Section 2 and Appendix A, the formal statement is that $\det U, \det V \in \{1, -1\}$ and $UAV = \operatorname{diag}(1,1,1,1,2,2)$; moreover, no integer 6×6 matrices P, Q with determinants in $\{1, -1\}$ satisfy $PAQ = \operatorname{diag}(1,1,1,1,1,4)$. For the literal integer matrices A_{52}, U_{52}, V_{52} of Appendix A, it states that U_{52}, V_{52} have determinants in $\{1, -1\}$ and give the diagonalization in Proposition 2. For every $\mu \in \mathbb{Q}$, there is a nonzero $v \in \mathbb{Q}^{14}$ with $A_{52}v = \mu v$ if and only if $\mu = 0$ or 4. For all integer 14×14 matrices P, Q with determinants in $\{1, -1\}$ and integer vectors d, the equality $PA_{52}Q = \operatorname{diag}(d)$ implies that some d_i is divisible by 2 and not by 4. Separate formal identities give $A_{52}(A_{52} - 4I) = 0$ and characteristic polynomial $t^5(t-4)^9$, where t is an indeterminate. Two independent certificate checkers construct both matrices from tabloids and standard polytabloids: column j records the coordinates of $X_k e_{T_j}$

in the stated ordered basis. They compare the resulting matrices with these very literals and compute the conjectured lists. This identifies the formal matrices with the Specht-module actions. That identification and the application of Conjecture 7.4's formula to obtain the prediction lists are outside Lean.

For every natural $n \geq 3$, Lean proves both group-ring identities of Lemma 5 in $\mathbb{Z}\mathfrak{S}_n$. Formal permutations are numbered from 0; increasing each letter by one gives the paper's transpositions. For any finite index set, integer square matrices A, B, U, V and integer vector d, Lean also proves that

$$AB = 2I, \quad \det U, \det V \in \{1, -1\}, \quad UAV = \operatorname{diag}(d)$$

imply that each $d_i \in \{1, -1, 2, -2\}$ and the number divisible by 2 is $v_2(|\det A|)$. The formal entries may be signed and unordered; the paper normalizes signs and divisibility order. Finally, for natural a, b, f with $a + b \leq f$, Lean proves that all the factors $2^{[r\leq a]+[r\leq b]}$, $1 \leq r \leq f$, equal 2 for $r \leq a + b$ and 1 otherwise if and only if $a = 0$ or $b = 0$. Here $a = N_{(1,3)}$, $b = N_{(3,1)}$, $f = f^{\lambda}$, and r counts from the largest factor. The passage to lattices, the eigenvalue theorem, the tableau counts and the extension argument remain the written steps in Theorem 3 and Corollary 4.

The finite computation for $n \leq 5$ in Appendix C is checked independently by a separate program; its argument for every X_k-preserved lattice when $n \leq 4$ is written. These two small-case statements are outside Lean. The programs also reproduce Stanley's example $\lambda = (5, 1)$, $k = 5$, with Smith form $(1, 1, 3, 3, 12)$ [9, example following Conjecture 7.4]. Certificate checking took about two seconds in total. The supplementary material accompanying this paper contains the checking notes and README. The checking notes give the full formal statements and declaration names, their logical dependencies, the supporting enumeration for $n \leq 8$, and the software and hardware details; reproduction instructions are in the README.

Appendix A Unimodular transformations

These matrices give the unimodular diagonalizations used in Sections 2 and 3. Let $B = 2A^{-1}$; by Lemma 5 it is the matrix of $X_3 - (1, 2)$ in the same basis, and it is integral. The auxiliary matrix is

$$B = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & -2 & 0 & 0 & 0 & 0 \\ 1 & 0 & -2 & 0 & 0 & 0 \\ 1 & 2 & 0 & 2 & 0 & 0 \\ -1 & 0 & 2 & 0 & 2 & 0 \\ 0 & -1 & -1 & -1 & -1 & -1 \end{pmatrix},$$

and $AB = BA = 2I$.

With

$$U = \begin{pmatrix} 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & -1 & 0 & 0 & 0 \\ 0 & -1 & -1 & -1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 \\ -1 & 0 & 2 & 0 & 2 & 0 \\ 0 & -1 & -1 & -1 & -1 & -1 \end{pmatrix}, \qquad V = \begin{pmatrix} 1 & -1 & 1 & 1 & -1 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

one has $\det U = -1$, $\det V = 1$ and $UAV = \mathrm{diag}(1,1,1,1,2,2)$, which proves Theorem 1. For the ordered basis in Section 3, the second matrix is

$$A_{52} = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 1 & 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 & 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 1 & 4 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 4 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 4 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 4 & 0 & 0 \\
1 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 4 & 0 \\
1 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 4
\end{pmatrix}.$$

The transformations are

$$U_{52} = \begin{pmatrix}
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & -1 & 0 & -1 & 1 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 1 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 3 & -3 & 0 & 0 & 1 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 4 & -5 & 1 & 0 & 1 & -2 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix},$$

$$V_{52} = \begin{pmatrix}
1 & -1 & -1 & 0 & -1 & 2 & 0 & -2 & 0 & -4 & 4 & 2 & -2 & 0 \\
0 & 1 & 1 & 0 & 1 & -2 & 0 & -2 & 0 & -8 & 8 & 2 & -2 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 3 & -3 & -1 & 1 & 0 \\
0 & 0 & 1 & 0 & 1 & -2 & -4 & -2 & -4 & -24 & 24 & 6 & -10 & 4 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 7 & -7 & -2 & 3 & -1 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 8 & -8 & -2 & 3 & -1 \\
0 & 0 & 0 & 0 & 1 & -2 & 0 & 2 & 0 & 4 & -8 & -2 & 2 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 5 & -4 & -1 & 2 & -1 \\
0 & 0 & 0 & 1 & -3 & 6 & 0 & -6 & 0 & -12 & 12 & 2 & -6 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}.$$

Their determinants are units and $U_{52}A_{52}V_{52}$ is the diagonal in Proposition 2.

Appendix B A character formula

A character formula gives another expression for the multiplicity m. Let L be the lattice of Theorem 3, $c = (1,3)(2,3)$, $\mathbb{Z}_{(2)}$ is obtained from $\mathbb{Z}$ by inverting the odd integers, and $L_{(2)} = \mathbb{Z}_{(2)} \otimes_{\mathbb{Z}} L$.

The expression $2(1,3)$ means twice the operator induced by the transposition exchanging 1 and 3; $\chi^\lambda(g)$ is the trace of the permutation g on $S^\lambda_{\mathbb{Q}}$.

Remark 6 (proved in writing; not machine-checked). Over $\mathbb{Z}_{(2)}$, the idempotent $P = \frac{1}{3}(1 + c + c^2)$ gives the same conclusion in another form: $X_3^2 = 1 + 3P$, so X_3 is invertible on $(1-P)L_{(2)}$ and acts as $2(1,3)$ on $PL_{(2)}$. Comparing the two descriptions gives $m = (f^\lambda + 2\chi^\lambda((1,2,3)))/3$, where χ^λ is the character of $S^\lambda_{\mathbb{Q}}$.

Appendix C The cases with $n \leq 5$

On the integral Specht lattices, the computation for $n \leq 5$ and the argument for $n \leq 4$ establish the smallest possible size of a counterexample.

Proposition 7 (exact computation on the integral Specht lattices)**.** *On $S^\lambda_{\mathbb{Z}}$, Conjecture 7.4 holds for every $\lambda \vdash n \leq 4$ and every $1 \leq k \leq n$. For $n = 5$ it fails exactly for $\lambda = (3,1,1)$ with $k = 3$ and with $k = 5$.*

For $\lambda = (3,1,1)$ and $k = 5$ the Smith form is $(1,1,2,2,4,4)$ and the prediction $(1,1,1,4,4,4)$. For $n \leq 4$ there is also a written argument, which gives more.

Proposition 8 (proved in writing; not machine-checked)**.** *For every $\lambda \vdash n \leq 4$ and every $1 \leq k \leq n$, Conjecture 7.4 holds on every lattice in $S^\lambda_{\mathbb{Q}}$ preserved by X_k.*

For $S^\lambda_{\mathbb{Z}}$, this also follows from Proposition 7.

Proof (written). We use again that X_k acts diagonally on Young's seminormal basis, with eigenvalues the contents of the cells occupied by k in the standard tableaux [10, Theorem 5.8]. The determinant and the rational rank of $X_k - a$ ($a \in \mathbb{Z}$) therefore do not depend on the lattice.

If $k = 1$, then $X_1 = 0$, and 1 occupies $(1,1)$, of content 0, in every tableau; the prediction is $(0,\dots,0)$. If $k = 2$, then $X_2 = (1,2)$ is an involution, hence unimodular on every lattice, and its Smith form is $(1,\dots,1)$; the entry 2 occupies $(1,2)$ or $(2,1)$, of content ± 1, so every predicted factor is 1. If $f^\lambda = 1$, then $\lambda = (n)$ or (1^n), X_k is the scalar $\pm(k-1)$, and k occupies $(1,k)$ or $(k,1)$, as predicted. The remaining cases are listed below, with the cells occupied by k in the standard tableaux (multiplicities as exponents) and the prediction.

λ, k	cells of k	eigenvalues	prediction
$(2,1)$, 3	$(1,2)$, $(2,1)$	$1, -1$	$(1,1)$
$(2,2)$, 3	$(1,2)$, $(2,1)$	$1, -1$	$(1,1)$
$(2,2)$, 4	$(2,2)^2$	$0, 0$	$(0,0)$
$(3,1)$, 3	$(1,3)$, $(1,2)$, $(2,1)$	$2, 1, -1$	$(1,1,2)$
$(2,1,1)$, 3	$(3,1)$, $(1,2)$, $(2,1)$	$-2, 1, -1$	$(1,1,2)$
$(3,1)$, 4	$(1,3)^2$, $(2,1)$	$2, 2, -1$	$(1,2,2)$
$(2,1,1)$, 4	$(3,1)^2$, $(1,2)$	$-2, -2, 1$	$(1,2,2)$

In the first two rows $|\det X_k| = 1$, so the Smith form is $(1,1)$. In the third, X_4 is diagonalizable with both eigenvalues 0, so $X_4 = 0$. In the fourth and fifth, $|\det X_3| = 2$ is squarefree, which forces

$(1,1,2)$ (this also follows from Corollary 4). In the last two, $\lambda = (3,1)$ or $(2,1,1)$ with $k = 4$, we have $|\det X_4| = 4$. On a lattice preserved by X_4, the matrix of $X_4 - 2I$ for $(3,1)$, or of $X_4 + 2I$ for $(2,1,1)$, is an integer matrix of rational rank 1, since the eigenvalue ± 2 has multiplicity 2 and X_4 is diagonalizable; all its 2×2 minors vanish, so it has rank at most 1 over $\mathbb{F}_2$. As $X_4 \equiv X_4 \mp 2I \pmod 2$, at least two invariant factors are even, and the product 4 forces $(1,2,2)$. Every case agrees with the prediction. □

Appendix D An exterior-square calculation

This alternative calculation gives the same Smith form as the determinant and rank argument in Section 2. This derivation is written and not machine-checked.

The group-ring identity also gives the same conclusion through $AB = 2I$: every invariant factor divides 2, and $v_2(\det A) = 2$ of them equal 2. This is also the case $m = 2$ (T_1 and T_6) of Theorem 3.

By Theorem 3 (proved in writing), the same Smith form arises on every lattice stable under the permutations of $\{1,2,3\}$ in a module isomorphic to $S^{(3,1,1)}_{\mathbb{Q}}$.

The matrix can also be obtained from the exterior square of the standard representation; Λ^2 denotes the second exterior power. Let $e_1, \ldots, e_5$ be the standard basis of $\mathbb{Z}^5$, with $\mathfrak{S}_5$ permuting it, and let the root lattice $A_4 = \{x \in \mathbb{Z}^5 : \sum_i x_i = 0\}$. The map φ sending the tabloid with x in row 2 and y in row 3 to $e_x \wedge e_y \in \Lambda^2\mathbb{Z}^5$ is $\mathfrak{S}_5$-equivariant. If T has first column (c_1, c_2, c_3), then C_T is the symmetric group on $\{c_1, c_2, c_3\}$, and summing the six signed terms gives

$$\varphi(e_T) = 2\,(e_{c_2} - e_{c_1}) \wedge (e_{c_3} - e_{c_1}) \in 2\Lambda^2 A_4.$$

For standard T the first column is $(1, a, b)$ with $2 \le a < b \le 5$, each pair occurring once, so the six standard polytabloids are sent to $2f_a \wedge f_b$, where $f_i = e_i - e_1$. These form twice a $\mathbb{Z}$-basis of $\Lambda^2 A_4$. For the rational irreducibility of $S^{(3,1,1)}_{\mathbb{Q}}$, we use Grinberg's Corollary 5.13.3 [3]. Since $S^{\lambda}_{\mathbb{Q}}$ is irreducible and $\varphi \neq 0$, the map is injective on it, and $\frac{1}{2}\varphi$ is an $\mathfrak{S}_5$-equivariant isomorphism $S^{(3,1,1)}_{\mathbb{Z}} \cong \Lambda^2 A_4$. Now use the basis $b_i = e_i - e_5$ $(1 \le i \le 4)$ of A_4 and write $b_{ij} = b_i \wedge b_j$. The transpositions $(1,3)$ and $(2,3)$ fix b_4 and permute b_1, b_2, b_3, so X_3 preserves the span of b_{12}, b_{13}, b_{23} and the span of b_{14}, b_{24}, b_{34}, with matrices

$$\begin{pmatrix} 0 & 1 & -1 \\ 1 & -1 & 0 \\ -1 & 0 & -1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix}.$$

Their determinants are 2 and -2, and each has a 2×2 minor equal to ± 1, so each has Smith form $(1,1,2)$. Together they give $(1,1,1,1,2,2)$ again.

References

[1] A. Chervov, *Conjectures on Smith Normal Form for Jucys-Murphy elements (Stanley, Grinberg) - status and generalisations?*, with comments by D. Grinberg (comments 1262855 and 1262856), MathOverflow question 484782, December 25, 2024, https://mathoverflow.net/questions/484782.

[2] L. de Moura and S. Ullrich, The Lean 4 theorem prover and programming language, in *Automated Deduction – CADE 28*, Lecture Notes in Comput. Sci. 12699, Springer, 2021, 625–635.

[3] D. Grinberg, *An introduction to the symmetric group algebra*, Math 701, Spring 2024 lecture notes, June 12, 2026 version, https://darijgrinberg.gitlab.io/t/24s/sga.pdf.

[4] G. D. James, *The Representation Theory of the Symmetric Groups*, Lecture Notes in Math. 682, Springer, Berlin, 1978.

[5] A.-A. A. Jucys, Symmetric polynomials and the center of the symmetric group ring, *Rep. Math. Phys.* **5** (1974), 107–112.

[6] The mathlib Community, The Lean mathematical library, in *Proceedings of the 9th ACM SIGPLAN International Conference on Certified Programs and Proofs (CPP 2020)*, ACM, 2020, 367–381.

[7] G. E. Murphy, A new construction of Young's seminormal representation of the symmetric groups, *J. Algebra* **69** (1981), 287–297.

[8] B. E. Sagan, *The Symmetric Group: Representations, Combinatorial Algorithms, and Symmetric Functions*, 2nd ed., Grad. Texts in Math. 203, Springer, New York, 2001.

[9] R. P. Stanley, Smith normal form in combinatorics, *J. Combin. Theory Ser. A* **144** (2016), 476–495; arXiv version 2, Section 7, page 15; arXiv:1602.00166.

[10] A. M. Vershik and A. Yu. Okounkov, *A new approach to the representation theory of the symmetric groups. II*, arXiv:math/0503040v1, 2005, https://arxiv.org/abs/math/0503040v1.