A conjecture of Elgin, Reading and Stella on rank-two scattering coefficients

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Abstract

For the rank-two cluster scattering diagram with initial walls $1+x$ and $1+y^5$, we prove $\tau^{1,5}(2j,j) = \frac{1}{j}\sum_{\ell}\binom{\ell}{j-\ell+1}\binom{j+\ell-1}{\ell}$ for every $j \geq 1$, in the convention of Burcroff and coauthors and of Elgin, Reading and Stella's table and code. This proves Elgin, Reading and Stella's Conjecture 13. Elgin, Reading and Stella provided extensive computations and stated eighteen conjectures on rank-two cluster scattering coefficients. Burcroff, Lee, Mou, Musiker and Reineke proved Conjectures 1–3, 5–10 and 14–18, leaving Conjecture 13 unresolved and "mysterious from both the quiver moduli and tight grading points of view." Kontsevich and Soibelman provided ordered factorization, and Gross, Hacking, Keel and Kontsevich provided the change of lattice used here. The binomial sum was already recorded by Vladimir Kruchinin, with functional equations by Paul D. Hanna; Burcroff and coauthors identified its algebraicity and growth rate conditionally on the conjecture. We identify these numbers with the scattering coefficients. We observe that OpenAI's closely related quantum scattering framework admits the same five-and-one incoming multiplicities; whether it implies the coefficient formula or cubic remains unresolved. After a change of lattice and a monomial substitution, the fifth power of the wall function becomes the diagonal factor of an explicit Poisson automorphism. A grading separates that factor from the other walls and yields a differential equation with a unique algebraic solution. The coefficient identity is formalized in Lean from the definitions.

This work has not yet been reviewed by experts. The proofs, the Lean formalization and the text were produced by AI models (from OpenAI and Anthropic), working under the author's direction.

1 Introduction

For positive integers b, c, let $\mathfrak{D}(1+x^b, 1+y^c)$ be the consistent rank-two scattering diagram. A wall carries a power series. Following [5, (2.2)], $\tau^{b,c}(kd, ke)$ is its coefficient of $x^{kdb}y^{kec}$ on the ray proportional to (db, ec), for coprime $d, e \geq 1$; Section 3 gives the precise definitions. Put $S(t) = 1 + \sum_{j\geq 1}\tau^{1,5}(2j,j)t^j \in \mathbb{Q}[\![t]\!]$. Thus the function on the ray $(2,5)$ is $S(x^2y^5)$.

Theorem 1.1. *For every $j \geq 1$,*

$$\tau^{1,5}(2j,j) = \frac{1}{j}\sum_{\ell=0}^{\infty}\binom{\ell}{j-\ell+1}\binom{j+\ell-1}{\ell}, \tag{1.1}$$

where only the terms with $\lceil (j+1)/2\rceil \leq \ell \leq j+1$ are nonzero. Equivalently, $S = (1+\xi)/(1-\xi-\xi^2)$, where $\xi \in t\mathbb{Q}[\![t]\!]$ is the unique power series with $\xi = t+\xi^2+\xi^3$; and S is the unique power series with constant term 1 satisfying

$$t^2S^3 - 2tS^2 + (1+t)S - (1+t) = 0. \tag{1.2}$$

This proves Elgin, Reading and Stella's Conjecture 13 [6, Conjecture 13], posed in their first preprint on April 17, 2024. We follow their table and code [7], as does [5, (2.2)]; Remark B.2 quotes the printed definition and explains the convention.

OpenAI's closely related quantum scattering framework includes the case in which "simple incoming data are replaced by allowed products of elementary units" [15, Proposition 4.2]. Its elementary units include $D_p = E_{p,-1} = \prod_{k\geq 0}(1+v^{-1-2k}X_p)$ for a simple lattice root p, with quantum parameter v and torus monomial X_p [15, (3.2), Theorem 3.1 and §4.1]. "Occurrences, including repeated units, are counted separately" [15, §4.2]. Five D_{e_1} and one D_{e_2} match the multiplicities of our changed-lattice incoming data $(1+t)^5, 1+t$. The manuscript is dated September 24, 2026; our first version is dated October 22, 2026[1]. The classical passage remains unverified; implication of our all-j formula or cubic remains unresolved.

Thomas Elgin, Nathan Reading and Salvatore Stella [6] supplied the conjectures, high-degree symbolic computations, data and code [7]. Amanda Burcroff, Kyungyong Lee, Lang Mou, Gregg Musiker and Markus Reineke [5, Theorem 1.4] proved Conjectures 1–3, 5–10 and 14–18 of [6], using the change of lattice between $\mathfrak{D}(1+x^b, 1+y^c)$ and $\mathfrak{D}((1+x)^c, (1+y)^b)$ (from [9, Example 1.15]), Euler characteristics of quiver moduli through the refined GW/Kronecker correspondence of Reineke and Weist [16], and Burcroff, Lee and Mou's tight gradings [4]. They listed Conjectures 4, 12 and 13 as unsolved and 13 as "mysterious from both the quiver moduli and tight grading points of view" [5, §1.3]. Ryota Akagi [1, Proposition 1.3] proved Conjectures 1, 2, 3, 5, 6, 11 and 15 (proof of 11 due to Burcroff, Lee and Mou [1, Remark 1.4]), with results related to 14 and 16–18. Akagi's earlier low-degree forms [2] (first posted September 27, 2023) and Amanda Burcroff and Kyungyong Lee's *Broken lines and compatible pairs for rank 2 quantum cluster algebras* [3] are nearby work; nothing from either is used here. The ordered factorization is due to Kontsevich and Soibelman [12, Theorem 6], [13, Factorization Property]; we use the formulation of Gross, Pandharipande and Siebert [10, §1.3].

Vladimir Kruchinin's OEIS entry A217596 (March 19, 2013) lists the binomial sum in (1.1) as $-\mathrm{A217596}(j+1)$ [14]. Its generating function is $A = 1 - tS_*$, where $S_* = (1+\xi)/(1-\xi-\xi^2)$ for the series ξ in Theorem 1.1. Paul D. Hanna's functional equation (March 19, 2013) $A = 1 - t/A - t^2/A^2$ gives $A^3 = A^2 - tA - t^2$, which Burcroff, Lee, Mou, Musiker and Reineke used to deduce algebraicity [5, §9.3]; substituting $A = 1 - tS$ gives (1.2). Muhammed Sefa Saydam's polygon-dissection interpretation (January 27, 2025) is described in Appendix B.1; nothing from it is used here. Theorem 1.1 identifies these numbers with the scattering coefficients. The entry also credits Bala's 2015 recurrence (Appendix D). Scattering diagrams were introduced by Kontsevich and Soibelman [12] and by Gross and Siebert [11]; Gross, Hacking, Keel and Kontsevich provided the cluster-algebra construction [9]. The change of lattice used here is theirs [9, Example 1.15 and Appendix C]. We reuse the Kontsevich–Soibelman factorization, the change of lattice (which we reprove in the form needed) and the pentagon identity.

The coefficient theorem is formalized in Lean (Section 8).

2 Outline of the proof

To identify the sequence with the scattering coefficients, the hardest step is to obtain a closed equation for one wall function without knowing the other walls. A change of lattice and a monomial substitution identify S^5 with the wall function on $(1,1)$ in the ordered factorization of the fifth

[1] The public tag scattering-coefficients-v1 identifies our October 22 first version.

iterate of the automorphism T of $\mathbb{Q}[\![u,w]\!]$, where

$$T(u)=\frac{u(1+u+w)}{1+u},\qquad T(w)=\frac{w}{1+u}.$$

Since T fixes $u+w$, grading by exponent difference isolates three components of the transformed sum (Section 5). Factoring this expression and comparing logarithms yields two auxiliary series (Section 5.1). The Poisson bracket gives a differential equation; its unique solution and Lagrange inversion give the coefficients.

3 Scattering diagrams and the coefficients τ

We set up the diagram and factorization used below. We write *wall* for its wall-crossing automorphism, retaining the supporting ray and attached series. Routine algebraic verifications are collected in Appendix A.

3.1 Walls and ordered products

Let z_1, z_2 be variables and $\mathcal{A}=\mathbb{Q}[\![z_1,z_2]\!]$, with maximal ideal $\mathfrak{m}=(z_1,z_2)$ and the $\mathfrak{m}$-adic topology. For $a\in\mathbb{N}^2$ write $z^a=z_1^{a_1}z_2^{a_2}$ and $|a|=a_1+a_2$, and let $\omega(a,r)=a_1r_2-a_2r_1$. A *ray* is a primitive vector $r\in\mathbb{N}^2\setminus\{0\}$, that is, one with $\gcd(r_1,r_2)=1$; rays are ordered by their slope $r_2/r_1\in[0,\infty]$. Let $\Lambda=1+t\mathbb{Q}[\![t]\!]$ be the group of one-variable power series with constant term 1.

For a ray r and $f\in\Lambda$, the *wall* $\theta_{r,f}$ is the continuous ring automorphism of $\mathcal{A}$ with

$$\theta_{r,f}(z_1)=z_1\,f(z^r)^{r_2},\qquad \theta_{r,f}(z_2)=z_2\,f(z^r)^{-r_1},\qquad \text{so}\qquad \theta_{r,f}(z^a)=z^a f(z^r)^{\omega(a,r)}. \tag{3.1}$$

It is invertible, with $\theta_{r,f}^{-1}=\theta_{r,1/f}$, and walls on the same ray compose by $\theta_{r,f}\theta_{r,g}=\theta_{r,fg}$. We compose automorphisms as maps, $(\phi\psi)(\eta)=\phi(\psi(\eta))$. For example, $\theta_{(1,0),1+t}$ fixes z_1 and sends z_2 to $z_2/(1+z_1)$.

Since $f(z^r)-1\in\mathfrak{m}^{|r|}$, a wall satisfies $\theta_{r,f}(\eta)-\eta\in\mathfrak{m}^{|r|+1}$ for every $\eta\in\mathcal{A}$. Hence, modulo $\mathfrak{m}^N$, only the finitely many walls with $|r|<N$ act nontrivially. For a *family* $f=(f_r)$ of series $f_r\in\Lambda$, one for each ray, the *ascending product* $\prod_r^{\to}\theta_{r,f_r}$ is the automorphism of $\mathcal{A}$ that agrees, modulo $\mathfrak{m}^N$ and for every N, with the finite composition $\theta_{r^{(1)},f_{r^{(1)}}}\circ\theta_{r^{(2)},f_{r^{(2)}}}\circ\cdots$ over the rays with $|r|<N$ listed by increasing slope. Thus the ray of smallest slope is leftmost and is applied last.

3.2 Consistent diagrams

Consistency equates the initial-wall product with the ordered product.

Definition 3.1. Let $P_1,P_2\in\Lambda$. A family $f=(f_r)$ is *consistent for* $\mathfrak{D}(P_1,P_2)$ if

$$\theta_{(0,1),P_2}\circ\theta_{(1,0),P_1}=\prod_r^{\to}\theta_{r,f_r}, \tag{3.2}$$

where the product runs over all rays, including $(1,0)$ and $(0,1)$. Equivalently, for every N the two sides agree on z_1 and z_2 modulo $\mathfrak{m}^N$, the right side being the finite ascending product over $|r|<N$.

By Proposition 3.5, a consistent family exists and is unique. Its entries on the axes are $f_{(1,0)}=P_1$ and $f_{(0,1)}=P_2$: on the quotient $(z_2)/(z_2^2)$ every wall with $r_2\geq 1$ acts as the identity and $\theta_{(1,0),f}$ acts as multiplication by $f(z_1)^{-1}$, so (3.2) forces $f_{(1,0)}=P_1$, and symmetrically $f_{(0,1)}=P_2$. Its function f_r on a ray r with $r_1,r_2>0$ is the wall function on that ray of the diagram $\mathfrak{D}(P_1,P_2)$ of [5, §2] (Remark B.1).

Example 3.2. For $\mathfrak{D}(1+t,1+t)$ the consistent family has $1+t$ on the rays $(1,0)$, $(1,1)$, $(0,1)$ and 1 elsewhere: with $\mathbf{X}=\theta_{(1,0),1+t}$, $\mathbf{Y}=\theta_{(0,1),1+t}$ and $\mathbf{Z}=\theta_{(1,1),1+t}$, the identity $\mathbf{YX}=\mathbf{XZY}$ holds, both sides sending $z_1\mapsto z_1(1+z_2)$ and $z_2\mapsto z_2/(1+z_1+z_1z_2)$. This is the pentagon identity.

Definition 3.3. Let $b,c\geq 1$, let f be the consistent family for $\mathfrak{D}(1+t^b,1+t^c)$, and let $d,e\geq 1$ be coprime. If v is the ray proportional to (db,ec), then $f_v(z^v)$ is a series in $x^{db}y^{ec}$, where $x=z_1$, $y=z_2$ [5, §2], and the coefficients $\tau^{b,c}(kd,ke)$ are defined [5, (2.2)] by

$$f_v(z^v)=1+\sum_{k\geq 1}\tau^{b,c}(kd,ke)\,x^{kdb}y^{kec}.$$

For $(b,c)=(1,5)$ and $(d,e)=(2,1)$ the vector $(db,ec)=(2,5)$ is already a ray, so $\tau^{1,5}(2j,j)$ is the coefficient of t^j in $f_{(2,5)}(t)$, where f is the consistent family for $\mathfrak{D}(1+t,1+t^5)$. That is, $S=f_{(2,5)}$, the series of the introduction. This is the only coefficient family used below.

3.3 The Poisson bracket and ordered factorization

The Poisson bracket controls ordered factorization. On $\mathcal{A}=\mathbb{Q}[\![z_1,z_2]\!]$ let

$$\{f,g\}=(z_1\partial_{z_1}f)(z_2\partial_{z_2}g)-(z_2\partial_{z_2}f)(z_1\partial_{z_1}g),\qquad\text{so}\qquad\{z^a,z^b\}=\omega(a,b)\,z^{a+b}.$$

It is a continuous biderivation, and $\{\log\eta,g\}=\{\eta,g\}/\eta$ for $\eta\in 1+\mathfrak{m}$. An automorphism ϕ of $\mathcal{A}$ is *unipotent* if $\phi(z_i)\in z_i(1+\mathfrak{m})$ for $i=1,2$, and *Poisson* if $\phi\{f,g\}=\{\phi f,\phi g\}$ for all f,g.

Lemma 3.4. *1. A unipotent automorphism ϕ is Poisson if and only if $\{\phi(z_1),\phi(z_2)\}=\phi(z_1)\phi(z_2)$.*

2. Every wall is a unipotent Poisson automorphism, and so is every ascending product of walls.

Proposition 3.5 (Kontsevich–Soibelman)**.** *Let ϕ be a unipotent Poisson automorphism of $\mathcal{A}$. There is exactly one family $f=(f_r)$ with $\phi=\prod_r^{\rightarrow}\theta_{r,f_r}$. Moreover, any automorphism of $\mathcal{A}$ has at most one such factorization.*

By Lemma 3.4, the left side of (3.2) is unipotent and Poisson, so every $\mathfrak{D}(P_1,P_2)$ has exactly one consistent family.

4 The change of lattice

The change of lattice relates the ray $(2,5)$ of $\mathfrak{D}(1+t,1+t^5)$ to the ray $(2,1)$ of $\mathfrak{D}_5=\mathfrak{D}((1+t)^5,1+t)$. The general statement is the change of lattice of [5, Proposition 2.5], going back to [9, Example 1.15 and Appendix C]; we prove the case needed by a forward substitution.

Let $\mathfrak{D}_5$ live on $\mathbb{Q}[\![X,Y]\!]$ and $\mathfrak{D}(1+t,1+t^5)$ on $\mathbb{Q}[\![x,y]\!]$, and let $\kappa\colon\mathbb{Q}[\![X,Y]\!]\to\mathbb{Q}[\![x,y]\!]$ be the injective continuous ring homomorphism with $\kappa(X)=x$ and $\kappa(Y)=y^5$. For a ray (d,e) put

$$v(d,e)=\begin{cases}(d,5e) & \text{if } 5\nmid d,\\ (d/5,e) & \text{if } 5\mid d,\end{cases}\qquad G^{\kappa}_{(d,e)}(t)=\begin{cases}G(t)^{1/5} & \text{if } 5\nmid d,\\ G(t^5) & \text{if } 5\mid d,\end{cases}$$

for $G\in\Lambda$, where $G^{1/5}$ is the unique fifth root in Λ. The map $(d,e)\mapsto v(d,e)$ is a bijection of rays that multiplies slopes by 5, so it preserves their order.

Lemma 4.1. *For every ray (d,e) and $G\in\Lambda$, $\kappa\circ\theta_{(d,e),G}=\theta_{v(d,e),G^{\kappa}_{(d,e)}}\circ\kappa$.*

Proposition 4.2. *Let G be the consistent family for $\mathfrak{D}_5$ and f the consistent family for $\mathfrak{D}(1+t, 1+t^5)$. Then $f^5_{(2,5)} = G_{(2,1)}$.*

Proof. Define a family f' on $\mathbb{Q}[\![x,y]\!]$ by $f'_{v(d,e)} = (G_{(d,e)})^{\kappa}_{(d,e)}$. By Lemma 4.1, applied to each wall, and continuity of κ, $\kappa \circ \prod^{\to} \theta_{\rho,G_\rho} = \prod^{\to} \theta_{v,f'_v} \circ \kappa$. The initial walls behave the same way: $(1,0)$ goes to $(1,0)$ with $((1+t)^5)^{1/5} = 1+t$, and $(0,1)$ goes to $(0,1)$ with $1+t^5$. Hence, with $\Phi = \theta_{(0,1),1+t^5}\theta_{(1,0),1+t}$ on $\mathbb{Q}[\![x,y]\!]$, the consistency of G gives $\Phi \circ \kappa = \prod^{\to} \theta_{v,f'_v} \circ \kappa$. So Φ and $\Pi = \prod^{\to} \theta_{v,f'_v}$ agree on x and on y^5. Both send y into $y(1+\mathfrak{m})$, and a fifth root of unity in $1+\mathfrak{m}$ is 1, so they also agree on y. Thus f' is consistent for $\mathfrak{D}(1+t, 1+t^5)$, and $f = f'$ by Proposition 3.5. In particular $f_{(2,5)} = f'_{v(2,1)} = G^{1/5}_{(2,1)}$. □

4.1 The monomial substitution

A monomial substitution identifies $G_{(2,1)}$ with the diagonal factor of T^5.

Let $\iota\colon \mathbb{Q}[\![u,w]\!] \to \mathbb{Q}[\![X,Y]\!]$ be the injective continuous ring homomorphism with $\iota(u) = X$ and $\iota(w) = XY$. On exponents it is the linear map $L(p,q) = (p+q, q)$, which has determinant 1, so $\omega(La, Lr) = \omega(a,r)$. It maps rays to rays lying in the cone $0 \le q \le p$, and it maps slopes by $q/p \mapsto q/(p+q)$, an increasing map; in particular $L(1,0) = (1,0)$, $L(0,1) = (1,1)$ and $L(1,1) = (2,1)$. Checking on u and w as in Lemma 4.1, one finds for every ray r and $f \in \Lambda$ that

$$\iota \circ \theta_{r,f} = \theta_{Lr,f} \circ \iota. \tag{4.1}$$

Let $\theta_u = \theta_{(1,0),1+t}$ and $\theta_w = \theta_{(0,1),1+t}$ on $\mathbb{Q}[\![u,w]\!]$, and let $T = \theta_u \circ \theta_w$. With $\mathbf{X}, \mathbf{Y}, \mathbf{Z}$ as in Example 3.2 (on $\mathbb{Q}[\![X,Y]\!]$), (4.1) gives $\iota \circ T = \mathbf{XZ} \circ \iota$.

Proposition 4.3. *The automorphism T^5 of $\mathbb{Q}[\![u,w]\!]$ is unipotent and Poisson. Let φ be the family with $T^5 = \prod^{\to} \theta_{r,\varphi_r}$ (Proposition 3.5). Then the consistent family G for $\mathfrak{D}_5$ satisfies $G_{(2,1)} = \varphi_{(1,1)}$.*

Proof. The first sentence follows from Lemma 3.4. Let G' be the family on $\mathbb{Q}[\![X,Y]\!]$ with $G'_{Lr} = \varphi_r$ for every ray r, $G'_{(0,1)} = 1+t$, and $G' = 1$ on all other rays (those of slope in $(1,\infty)$). By (4.1), $\iota \circ T^5 = (\mathbf{XZ})^5 \circ \iota$ and $\iota \circ \prod^{\to} \theta_{r,\varphi_r} = \prod^{\to}_{\text{slope}\le 1} \theta_{\rho,G'_\rho} \circ \iota$. So $(\mathbf{XZ})^5$ and $\Pi' = \prod^{\to}_{\text{slope}\le 1} \theta_{\rho,G'_\rho}$ agree on X and on XY; since both send X to X times a unit and $\mathbb{Q}[\![X,Y]\!]$ is a domain, they agree on Y, so $(\mathbf{XZ})^5 = \Pi'$. The pentagon identity $\mathbf{YXY}^{-1} = \mathbf{XZ}$ (Example 3.2) gives $\mathbf{YX}^5 = (\mathbf{XZ})^5\mathbf{Y} = \Pi' \circ \theta_{(0,1),1+t}$. The left side is $\theta_{(0,1),1+t} \circ \theta_{(1,0),(1+t)^5}$, and the right side is the ascending product of the family G', the wall of slope ∞ being rightmost. So G' is consistent for $\mathfrak{D}_5$, $G = G'$ by Proposition 3.5, and $G_{(2,1)} = G'_{L(1,1)} = \varphi_{(1,1)}$. □

Lemma 4.4. *The automorphism T is given by*

$$T(u) = \frac{u(1+u+w)}{1+u}, \qquad T(w) = \frac{w}{1+u}.$$

Hence, with $s = u+w$, one has $T(s) = s$ and $T(u)/T(w) = (1+s)\,u/w$.

From now on we write $F = \varphi_{(1,1)} \in \Lambda$, the diagonal factor of T^5. Propositions 4.2 and 4.3 give

$$S^5 = f^5_{(2,5)} = F. \tag{4.2}$$

Figure 1 shows these rays in low degree.

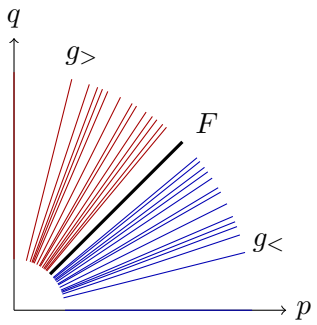


Figure 1: The rays $r = (p, q)$ whose function in the ascending factorization of T^5 has a nonzero coefficient in total degree at most 12 (that is, $[t^j]\varphi_r \neq 0$ for some $j \geq 1$ with $j|r| \leq 12$), computed by the construction in the proof of Proposition 3.5. Blue: slope below 1, the factors of $g_<$ ($h < 0$). Red: slope above 1, the factors of $g_>$ ($h > 0$). Thick black: the diagonal, whose function is F. Each ray is drawn from radius 0.9 to 4.2 for legibility. The proof uses only the sign of h on each side of the diagonal.

5 The diagonal factor

The grading by the difference of the exponents separates the two sides of the diagonal. Throughout, $A = \mathbb{Q}[\![u, w]\!]$ and $t = uw$; we identify $\mathbb{Q}[\![t]\!]$ with the subring of A of series in uw.

For a monomial put $h(u^p w^q) = q - p$, and for $k \in \mathbb{Z}$ let π_k be the continuous projection of A onto the closed span of the monomials with $h = k$. The monomials with $h = 0$ are the powers of t, so $\pi_0(A) = \mathbb{Q}[\![t]\!]$, and $\pi_{-1}(A) = u\mathbb{Q}[\![t]\!]$, $\pi_1(A) = w\mathbb{Q}[\![t]\!]$. Let $\mathcal{O}_{\geq k}$ and $\mathcal{O}_{\leq k}$ be the closed subspaces of series all of whose monomials have $h \geq k$, respectively $h \leq k$. Since h is additive, $\mathcal{O}_{\geq 0}$ is a closed subring, $\mathcal{O}_{\geq k}\mathcal{O}_{\geq l} \subseteq \mathcal{O}_{\geq k+l}$, and likewise for $\mathcal{O}_{\leq}$; in particular $1 + \mathcal{O}_{\geq 1}$ and $1 + \mathcal{O}_{\leq -1}$ are groups under multiplication.

Group the factorization of T^5 by slope: let $g_<$ be the ascending product of the walls θ_{r,φ_r} with $r_2 < r_1$ (including $(1, 0)$), let $g_>$ be that of the walls with $r_2 > r_1$ (including $(0, 1)$), and let $\theta_F = \theta_{(1,1),F}$, so that

$$T^5 = g_< \circ \theta_F \circ g_>, \qquad \theta_F(u) = uF(t), \quad \theta_F(w) = w/F(t). \tag{5.1}$$

The automorphisms $g_<$ and $g_>$ are Poisson, by Lemma 3.4. We show that $g_<^{-1}(s)$ has only three graded components.

Here and below $z^a = u^{a_1} w^{a_2}$, and $s = u + w$. Let $\xi \in t\mathbb{Q}[\![t]\!]$ be the unique series with $\xi(1 - \xi - \xi^2) = t$, equivalently $\xi = t + \xi^2 + \xi^3$, and let

$$S_* = \frac{1 + \xi}{1 - \xi - \xi^2} = \frac{1}{t} - \frac{1}{\xi}.$$

(The second expression holds because $1/\xi = (1 - \xi - \xi^2)/t$.) Since $\xi = t + O(t^2)$, the substitution $t \mapsto \xi$ identifies $\mathbb{Q}[\![t]\!]$ with $\mathbb{Q}[\![\xi]\!]$.

The following theorem, proved in Section 7, is the core of the argument.

Theorem 5.1. *Let $F \in \Lambda$, let θ_F be as in (5.1), and let $g_<, g_>$ be Poisson automorphisms of $A = \mathbb{Q}[\![u, w]\!]$ such that, for every $a \in \mathbb{N}^2$,*

$$g_>(z^a) \in z^a(1 + \mathcal{O}_{\geq 1}), \qquad g_<^{-1}(z^a) \in z^a(1 + \mathcal{O}_{\leq -1}), \tag{5.2}$$

and such that $T^5 = g_< \circ \theta_F \circ g_>$. *Then* $F = S_*^5$, *where* $S_* = (1+\xi)/(1-\xi-\xi^2)$ *and* $\xi \in t\mathbb{Q}[\![t]\!]$ *is the unique series with* $\xi = t + \xi^2 + \xi^3$.

Lemma 5.2. *The factors in* (5.1) *satisfy* (5.2), *and* θ_F *commutes with every projection* π_k.

Lemma 5.3. *Under the hypotheses of Theorem 5.1, the series* $P = g_<^{-1}(s)$, $s = u + w$, *has the form*

$$P = uF(t) + \beta(t) + w \qquad \text{for some } \beta \in t\mathbb{Q}[\![t]\!].$$

Proof. By Lemma 4.4, $T^5(s) = s$, so $P = g_<^{-1}T^5(s) = \theta_F g_>(s)$. Write $g_>(u) = u(1+b_1)$ and $g_>(w) = w(1+b_2)$ with $b_i \in \mathcal{O}_{\geq 1}$. Then

$$P = \theta_F g_>(s) = uF\,\theta_F(1+b_1) + wF^{-1}\theta_F(1+b_2),$$

where $uF\theta_F(b_1) \in \mathcal{O}_{\geq 0}$ and the last term lies in $\mathcal{O}_{\geq 1}$, by Lemma 5.2. So $\pi_{-1}(P) = uF$ and $\pi_k(P) = 0$ for $k \leq -2$. On the other side, $g_<^{-1}(u) = u(1+\eta_1)$ and $g_<^{-1}(w) = w(1+\eta_2)$ with $\eta_i \in \mathcal{O}_{\leq -1}$, so $P = u(1+\eta_1) + w(1+\eta_2)$ has $\pi_1(P) = w$ and $\pi_k(P) = 0$ for $k \geq 2$. Hence $P = uF + \pi_0(P) + w$, and $\beta = \pi_0(P) \in \mathbb{Q}[\![t]\!]$ has no constant term because $P \in \mathfrak{m}$. □

5.1 Splitting the logarithm

Comparing graded components turns the factorization into two identities, one for F and one for $g_<$. For a unipotent automorphism ϕ of A let

$$\mu_\phi = \frac{\phi(u)/u}{\phi(w)/w} \in 1 + \mathfrak{m}.$$

Since $\phi\psi(u)/u = \phi(\psi(u)/u) \cdot \phi(u)/u$, and likewise for w, these ratios satisfy the cocycle rule $\mu_{\phi\psi} = \phi(\mu_\psi)\,\mu_\phi$.

Lemma 5.4. *Under the hypotheses of Theorem 5.1,*

$$(1+P)^5\,\mu_{g_<^{-1}} = F^2\,\theta_F(\mu_{g_>}), \tag{5.3}$$

with $\mu_{g_<^{-1}} \in 1 + \mathcal{O}_{\leq -1}$ *and* $\theta_F(\mu_{g_>}) \in 1 + \mathcal{O}_{\geq 1}$.

Proof. By Lemma 4.4, $\mu_T = 1 + s$, and since T fixes s, the cocycle rule gives $\mu_{T^5} = (1+s)^5$. Also $\mu_{\theta_F} = F^2$. Apply the cocycle rule to both sides of $g_<^{-1}T^5 = \theta_F g_>$: the left side gives $g_<^{-1}((1+s)^5)\,\mu_{g_<^{-1}} = (1+P)^5\mu_{g_<^{-1}}$, and the right side gives $\theta_F(\mu_{g_>})\,F^2$. The two memberships follow from (5.2), applied to u and w, and Lemma 5.2. □

Lemma 5.5. *Let* $F \in \Lambda$ *and* $\beta \in t\mathbb{Q}[\![t]\!]$. *There are* $\rho, \delta \in \Lambda$ *with* $\rho\delta = F$ *and* $\rho + t\delta = 1 + \beta$. *For such* ρ, δ,

$$1 + uF(t) + \beta(t) + w = (1 + u\delta)\cdot\rho\cdot(1 + w/\rho) \qquad \text{in } A. \tag{5.4}$$

Proof. The equation $\rho^2 - (1+\beta)\rho + tF = 0$ has a root $\rho \in \Lambda$: its coefficient of t^k is ρ_k plus a polynomial in $\rho_1, \ldots, \rho_{k-1}$, so the coefficients are determined recursively. Put $\delta = F/\rho$; then $\rho + t\delta = (\rho^2 + tF)/\rho = 1 + \beta$. Finally $(1+u\delta)(\rho + w) = \rho + w + u\rho\delta + uw\delta = 1 + \beta + uF + w$. □

Proposition 5.6. *Under the hypotheses of Theorem 5.1, let* β *be as in Lemma 5.3 and* ρ, δ *as in Lemma 5.5. Then*

$$F^2 = \rho^5, \qquad \delta^2 = \rho^3, \qquad \mu_{g_<^{-1}} = (1+u\delta)^{-5}.$$

Proof. The logarithm is a continuous isomorphism from the multiplicative group $1+\mathfrak{m}$ onto the additive group $\mathfrak{m}$, and it maps $1+\mathcal{O}_{\le -1}$ into $\mathcal{O}_{\le -1}$ and $1+\mathcal{O}_{\ge 1}$ into $\mathcal{O}_{\ge 1}$, since these are closed and stable under products. Take logarithms in (5.3), using Lemma 5.3 and (5.4):

$$5\log(1+u\delta)+5\log\rho+5\log(1+w/\rho)+\log\mu_{g_<^{-1}}=2\log F+\log\theta_F(\mu_{g_>}).$$

Here $u\delta\in\mathcal{O}_{\le -1}$ and $w/\rho\in\mathcal{O}_{\ge 1}$, so the terms lie in $\mathcal{O}_{\le -1}$, $\mathbb{Q}[\![t]\!]$, $\mathcal{O}_{\ge 1}$, $\mathcal{O}_{\le -1}$, $\mathbb{Q}[\![t]\!]$ and $\mathcal{O}_{\ge 1}$ respectively. Applying π_0 gives $5\log\rho=2\log F$, that is, $F^2=\rho^5$; then $\rho^2\delta^2=F^2=\rho^5$ gives $\delta^2=\rho^3$. Applying $\sum_{k<0}\pi_k$ gives $5\log(1+u\delta)+\log\mu_{g_<^{-1}}=0$. □

6 The differential equation

The Poisson property of $g_<$ gives a differential equation for (ρ,δ). Coefficient induction will show uniqueness. Let $E=u\partial_u+w\partial_w$ be the Euler derivation of A. For $G\in\mathbb{Q}[\![t]\!]$ write $G'=t\,dG/dt$. One checks directly that

$$E(G)=2G',\qquad \{G,u\}=-uG',\qquad \{G,w\}=wG',\qquad \{u,w\}=uw. \tag{6.1}$$

Lemma 6.1. *Let ϕ be a unipotent Poisson automorphism of A. Then for all $y\in A$,*

$$\phi(E(y))=E(\phi(y))+\{\log\mu_\phi,\phi(y)\}.$$

Formally, E is the Hamiltonian derivation of $\log u-\log w$, and Lemma 6.1 says that ϕ shifts $\log(u/w)$ by $\log\mu_\phi$; the lemma makes this precise without logarithms of u and w.

The following Poisson identity, called the string equation here, relates the Euler derivation to the bracket.

Proposition 6.2 (String equation)**.** *Under the hypotheses of Theorem 5.1, with P as in Lemma 5.3 and δ as in Lemma 5.5,*

$$(1+u\delta)(E(P)-P)=5\,\{u\delta,P\}.$$

Proof. The inverse of a Poisson automorphism is Poisson. Apply Lemma 6.1 with $\phi=g_<^{-1}$ and $y=s$. Since $E(s)=s$ and $\phi(s)=P$, and $\log\mu_\phi=-5\log(1+u\delta)$ by Proposition 5.6, it gives $P=E(P)-5\{u\delta,P\}/(1+u\delta)$. □

Lemma 6.3. *For $\rho,\delta\in\mathbb{Q}[\![t]\!]$ put $P=(\rho+w)(1+u\delta)-1$, and*

$$\Phi(\rho,\delta)=1-\rho+2\rho'-4t\delta-3t\delta',\qquad \Psi(\rho,\delta)=2\rho\delta'-3\delta\rho'.$$

Then $(1+u\delta)(E(P)-P)-5\{u\delta,P\}=(1+u\delta)\big(\Phi(\rho,\delta)+u\,\Psi(\rho,\delta)\big)$.

Proof. Expand the left side with the Leibniz rule and (6.1); a direct computation shows that it equals the right side. □

Corollary 6.4. *Under the hypotheses of Theorem 5.1, the series ρ and δ of Proposition 5.6 satisfy $\delta^2=\rho^3$ and*

$$1-\rho+2\rho'-4t\delta-3t\delta'=0. \tag{6.2}$$

Proof. By Lemma 5.3 and (5.4), $P=(\rho+w)(1+u\delta)-1$. Proposition 6.2 and Lemma 6.3 give $(1+u\delta)(\Phi+u\Psi)=0$. As $1+u\delta$ is a unit, $\Phi+u\Psi=0$; the term Φ has $h=0$ and $u\Psi$ has $h=-1$, so $\Phi=0$. □

Since $\delta = \rho^{3/2}$, equation (6.2) is the first-order equation $2\rho' - \rho + 1 - 3(t\rho^{3/2})' - t\rho^{3/2} = 0$ for ρ alone.

Lemma 6.5. *There is at most one pair $(\rho, \delta) \in \Lambda^2$ with $\delta^2 = \rho^3$ satisfying (6.2).*

Proof. Write $\rho = \sum \rho_k t^k$ and $\delta = \sum \delta_k t^k$, with $\rho_0 = \delta_0 = 1$. For $k \geq 1$ the coefficient of t^k in (6.2) is $(2k-1)\rho_k - (3k+1)\delta_{k-1}$. The coefficient of t^n in $\delta^2 = \rho^3$ reads $2\delta_n = 3\rho_n +$ (a polynomial in $\rho_1, \dots, \rho_{n-1}, \delta_1, \dots, \delta_{n-1}$). By induction, δ_{k-1} is determined by $\rho_1, \dots, \rho_{k-1}$, and then $\rho_k = (3k+1)\delta_{k-1}/(2k-1)$, since $2k - 1 \neq 0$. □

7 The solution and the proof of the theorem

An algebraic parametrization solves (6.2); Lagrange inversion then gives the binomial formula.

Lemma 7.1. *The pair $(\rho, \delta) = (S_*^2, S_*^3)$ lies in Λ^2, satisfies $\delta^2 = \rho^3$ and solves (6.2). Moreover S_* satisfies (1.2), and it is the only series in Λ that does.*

Proof. As $dt/d\xi = (1+\xi)(1-3\xi)$ is a unit, $G' = t\,(dG/d\xi)/(dt/d\xi)$ for $G \in \mathbb{Q}[\![t]\!]$, and

$$S_*' = \frac{\xi(2 + 2\xi + \xi^2)}{(1+\xi)(1-3\xi)(1-\xi-\xi^2)}.$$

With $(S_*^2)' = 2S_*S_*'$ and $(S_*^3)' = 3S_*^2S_*'$, the left side of (6.2) becomes $1 - S_*^2 + 4S_*S_*' - 4tS_*^3 - 9tS_*^2S_*'$, a rational function of ξ whose denominator is a product of the units $1+\xi$, $1-3\xi$, $1-\xi-\xi^2$; a direct computation shows that it is 0. Substituting $t = \xi(1-\xi-\xi^2)$ and S_* into (1.2) likewise gives 0. For uniqueness, the coefficient of t^k in (1.2) is S_k plus a polynomial in $S_0, \dots, S_{k-1}$. □

Proof of Theorem 5.1. By Corollary 6.4, Lemma 7.1 and Lemma 6.5, $\rho = S_*^2$ and $\delta = S_*^3$. Hence $F = \rho\delta = S_*^5$. □

Lemma 7.2. *For $j \geq 1$,*

$$[t^j]\, S_* = \frac{1}{j}\,[z^{j+1}]\,(1 - z - z^2)^{-j} = \frac{1}{j}\sum_{\ell} \binom{j+\ell-1}{\ell}\binom{\ell}{j+1-\ell}.$$

Proof. The series ξ satisfies $\xi = t\,R(\xi)$ with $R(z) = (1-z-z^2)^{-1}$. Lagrange inversion for integer powers [8, Theorem 2.1.1 and (2.2.1)] gives $[t^n]\,\xi^k = \frac{k}{n}[z^{n-k}]R(z)^n$ for all integers k and $n \neq 0$. With $k = -1$ and $n = j$, and since $S_* = 1/t - 1/\xi$, this gives the first equality. For the second, expand $(1 - z(1+z))^{-j} = \sum_{\ell \geq 0} \binom{j+\ell-1}{\ell} z^\ell (1+z)^\ell$ and take the coefficient of z^{j+1}. □

Proof of Theorem 1.1. By Lemma 5.2, the factorization (5.1) of T^5 satisfies the hypotheses of Theorem 5.1, so $F = S_*^5$. By (4.2), $S^5 = S_*^5$. Fifth roots in Λ are unique (if $G_1, G_2 \in \Lambda$ and $G_1^5 = G_2^5$, then $G_1/G_2 \in \Lambda$ is a fifth root of unity with constant term 1, hence $G_1/G_2 = 1$), so $S = S_*$. Lemma 7.2 gives (1.1); the term with index ℓ vanishes unless $0 \leq j + 1 - \ell \leq \ell$, that is, $\lceil (j+1)/2 \rceil \leq \ell \leq j+1$. Lemma 7.1 gives (1.2) and its uniqueness. □

8 Formal verification

Lean checks the coefficient theorem for the following formal definitions. Here `Slope` is $\mathbb{Q}_{\geq 0}$ with a top element for the vertical ray; $5/2$ represents $(2, 5)$. A family assigns a series in $\mathbb{Q}[\![t]\!]$ to each slope. The predicate `IsConsistentFamily` requires constant term 1 and Definition 3.1's ascending-product identity on both generators in every finite total degree. The expression `ersCoeff j` is the finite sum (1.1) over $\ell \leq j + 1$, with natural-number subtraction: none truncates for $j \geq 1$, and later terms vanish. The formal theorem `ErsConj13.ers_conjecture_13` states:

```
theorem ers_conjecture_13 :
    (∃ f, IsConsistentFamily (1 + PowerSeries.X) (1 + PowerSeries.X ^ 5) f) ∧
    ∀ f, IsConsistentFamily (1 + PowerSeries.X) (1 + PowerSeries.X ^ 5) f →
      ∀ j, 1 ≤ j → PowerSeries.coeff j (f ((5 / 2 : ℚ≥0) : Slope)) = ersCoeff j
```

The first conjunct gives existence; the second is Theorem 1.1's coefficient assertion for every consistent family and $j \geq 1$, in Definition 3.3's convention. Lean also proves the cubic identity and intermediate factorizations; the ξ-parametrization, uniqueness of the series in (1.2) and Lagrange inversion are written arguments (Appendix D).

The Lean proof uses Bala's recurrence in place of the written proof's Lagrange inversion. The public Lean guide, linked from the result's README, gives the logical dependencies, axiom list and build instructions. The scattering-diagram interpretation and convention comparison are outside Lean, as is Remark C.2. Checking relies on Lean's kernel and Mathlib.

Appendix A Ordered factorization and routine identities

The Poisson criterion and the degree-by-degree construction supply the ordered factorization used in the main argument.

Proof of Lemma 3.4. (1) By the chain rule, both $\phi\{f, g\}$ and $\{\phi f, \phi g\}$ equal $\sum_{i,j} \phi(\partial_i f)\phi(\partial_j g)$ times $\phi\{z_i, z_j\}$, respectively $\{\phi z_i, \phi z_j\}$, and $\{z_1, z_2\} = z_1 z_2$. (2) For $g = f(z^r)$ one has $z_i \partial_{z_i} g = r_i\, q\, g$ with $q = z^r f'(z^r)/f(z^r)$, and a direct computation gives $\{z_1 g^{r_2}, z_2 g^{-r_1}\} = z_1 g^{r_2}\, z_2 g^{-r_1} \big((1 + r_1 r_2 q)(1 - r_1 r_2 q) + r_1^2 r_2^2 q^2\big) = \theta_{r,f}(z_1)\, \theta_{r,f}(z_2)$, so (1) applies. Products of unipotent Poisson automorphisms, and their $\mathfrak{m}$-adic limits, are again unipotent and Poisson. □

Proof of Proposition 3.5. We give the degree-by-degree argument of [12] and [10, §1.3]. Say that two families agree below degree d if $[t^j] f_r = [t^j] f'_r$ whenever $j|r| < d$; then the corresponding ascending products agree on z_1, z_2 modulo $\mathfrak{m}^{d+1}$. If they agree below degree d, then, since every wall is the identity to first order, the coefficients of degree d enter additively at the lowest order:

$$\prod\nolimits^{\rightarrow} \theta_{r,f_r}(z_1) - \prod\nolimits^{\rightarrow} \theta_{r,f'_r}(z_1) \equiv z_1 \sum r_2\, (c_{r,j} - c'_{r,j})\, z^{jr},$$
$$\prod\nolimits^{\rightarrow} \theta_{r,f_r}(z_2) - \prod\nolimits^{\rightarrow} \theta_{r,f'_r}(z_2) \equiv -z_2 \sum r_1\, (c_{r,j} - c'_{r,j})\, z^{jr}$$

modulo $\mathfrak{m}^{d+2}$, where the sums run over the pairs (r, j) with $j|r| = d$ and $c_{r,j} = [t^j] f_r$. Distinct pairs give distinct monomials z^{jr}, and $(r_1, r_2) \neq 0$, so this difference vanishes only if the families agree below degree $d + 1$. This proves uniqueness.

For existence, suppose that $\Pi = \prod^{\rightarrow} \theta_{r,f_r}$ agrees with ϕ on z_1, z_2 modulo $\mathfrak{m}^{d+1}$, with $d \geq 1$, and write $\phi(z_1) - \Pi(z_1) \equiv z_1 \alpha$, $\phi(z_2) - \Pi(z_2) \equiv z_2 \alpha'$ modulo $\mathfrak{m}^{d+2}$, with α, α' homogeneous of degree d. Both ϕ and Π satisfy Lemma 3.4(1); subtracting the two identities and keeping the terms of degree $d + 2$ gives $z_1 z_2 (z_1 \partial_{z_1} \alpha + z_2 \partial_{z_2} \alpha') = 0$. So for each monomial z^k of degree d, the coefficients satisfy $k_1 \alpha_k + k_2 \alpha'_k = 0$, that is, $(\alpha_k, \alpha'_k) = c_k\, (r_2, -r_1)$ with $r = k/\gcd(k_1, k_2)$. Adding $c_k\, t^{\gcd(k_1,k_2)}$ to f_r for each such k changes the images of z_1, z_2 only in degrees $\geq d + 1$, and removes the discrepancy in degree $d + 1$. The resulting families converge coefficientwise, and their limit factors ϕ. □

Proof of Lemma 4.1. Both sides are continuous ring homomorphisms, so it suffices to compare them on X and Y. If $5 \nmid d$, then $v = (d, 5e)$, $z^v = \kappa(X^d Y^e)$ and $h = G^{1/5}$ give $\theta_{v,h}(x) = x\, h(z^v)^{5e} = \kappa(X\, G(X^d Y^e)^e)$ and $\theta_{v,h}(y^5) = y^5 h(z^v)^{-5d} = \kappa(Y\, G(X^d Y^e)^{-d})$. If $5 \mid d$, then $v = (d/5, e)$, $(z^v)^5 = \kappa(X^d Y^e)$ and $h(t) = G(t^5)$ give the same two values. □

Proof of Lemma 4.4. By (3.1), $\theta_w(u) = u(1+w)$, $\theta_w(w) = w$, $\theta_u(u) = u$ and $\theta_u(w) = w/(1+u)$. So $T(w) = w/(1+u)$ and $T(u) = \theta_u(u(1+w)) = u(1+\frac{w}{1+u})$. Adding the two values gives $T(s) = s$, and dividing them gives the last identity. □

Proof of Lemma 5.2. Let $\mathcal{G}_>$ be the set of unipotent automorphisms g with $g(z^a) \in z^a(1+\mathcal{O}_{\geq 1})$ for all a. A wall with $r_2 > r_1$ lies in $\mathcal{G}_>$, because $f(z^r)^{\omega(a,r)} - 1$ is a series in the monomials z^{jr}, $j \geq 1$, which have $h = j(r_2 - r_1) \geq 1$. If $g, g' \in \mathcal{G}_>$ and $g(z^a) = z^a(1+\eta)$ with $\eta \in \mathcal{O}_{\geq 1}$, then $g'g(z^a) = g'(z^a)\,(1+g'(\eta))$, and $g'(\eta) \in \mathcal{O}_{\geq 1}$ because g' maps each monomial z^k with $h(k) \geq 1$ into $z^k(1+\mathcal{O}_{\geq 1}) \subseteq \mathcal{O}_{\geq 1}$ and is continuous. So $\mathcal{G}_>$ is closed under composition, and it is closed under $\mathfrak{m}$-adic limits because its defining condition concerns coefficients only. Hence $g_> \in \mathcal{G}_>$. The same argument, with the inverse walls $\theta_{r,1/f}$, $r_2 < r_1$, taken in descending order, gives the statement for $g_<^{-1}$. Finally, $\theta_F(z^a) = z^a F(t)^{a_1 - a_2}$ has the same h as z^a in every term. □

Proof of Lemma 6.1. From the definition of the bracket, $uw\,E(y) = w\{u, y\} - u\{w, y\}$. Apply ϕ and write $\phi(u) = u\alpha$, $\phi(w) = w\alpha'$. Expanding $\{u\alpha, \phi y\}$ and $\{w\alpha', \phi y\}$ by the Leibniz rule and using the same identity for $\phi(y)$ gives $uw\alpha\alpha'\,\phi(Ey) = uw\alpha\alpha'(E(\phi y) + \{\alpha, \phi y\}/\alpha - \{\alpha', \phi y\}/\alpha')$. Divide by $uw\alpha\alpha'$, and note that $\{\alpha, g\}/\alpha - \{\alpha', g\}/\alpha' = \{\log(\alpha/\alpha'), g\}$. □

Appendix B Conventions and the associated sequence

The comparisons below fix the orientation and coefficient convention, and identify the sequence already recorded in the literature.

Remark B.1 (Path-ordered products and orientation). In [10, §1.3], crossing a wall supported on a line through r acts by $z^a \mapsto z^a f^{\langle n_0, a\rangle}$, where $n_0 \perp r$ is primitive with $\langle n_0, \gamma'\rangle > 0$ for the crossing path γ, and a diagram is consistent when the path-ordered product $\theta_\gamma = \theta_s \circ \cdots \circ \theta_1$ along every loop is the identity. A counterclockwise loop starting just below the positive x-axis crosses the positive x-axis, the rays $\mathbb{R}_{\geq 0} r$ with $r_1, r_2 > 0$ in order of increasing slope, and the positive y-axis, each by θ_{r,f_r}^{-1} in the notation (3.1), and then crosses the negative x- and y-axes by $\theta_{(1,0),P_1}$ and $\theta_{(0,1),P_2}$. So $\theta_\gamma = \mathrm{id}$ is the identity $\theta_{(0,1),P_2}\theta_{(1,0),P_1} = \theta_{(1,0),P_1} \circ (\prod_{r_1,r_2>0}^{\rightarrow} \theta_{r,f_r}) \circ \theta_{(0,1),P_2}$, which is (3.2). Burcroff, Lee, Mou, Musiker and Reineke draw the wall carrying f_v on $\mathbb{R}_{\leq 0} v$, in the sign convention of [11], in which z^a propagates in the direction $-a$ (see [10, §1.2]); the walls carry the same functions. As a check on the orientation, Definition 3.1 with $P_1 = P_2 = (1+t)^2$ gives $(1-t)^{-4}$ on the ray $(1,1)$ and $(1+t)^2$ on $(n+1, n)$ and $(n, n+1)$, as found in [10, §1.3].

Remark B.2 (Reading the definition in [6]). The convention of Definition 3.3 is that of the table of $\tau^{3,2}(i,j)$, $0 \leq i, j \leq 7$, printed in [6, §2], all of whose entries it reproduces, and of the code published with [6] (the method `wall_hom` of `scatcoef.py` in [7]); it is also the convention in which [5] restate the conjectures. The printed definition in [6, §2] says "$\zeta_1 = z_2^c$ and $\zeta_2 = z_1^{-b}$." Reading those substitutions literally interchanges b, c relative to the table/code convention: its coefficient $\tau^{b,c}(i,j)$ would be $\tau^{c,b}(i,j)$ in the convention used here. Under that reading $\tau^{3,2}(3,1)$ would be 1, whereas the table in [6] prints 0, which is its value under Definition 3.3; and Conjecture 13 would already fail at $j = 1$, since $\tau^{5,1}(2,1) = 0$. We therefore read [6] in the convention of its table and code.

B.1 The associated sequence

Here reversion means compositional inversion.

The right side of (1.1) was known as a sequence before the conjecture. Vladimir Kruchinin's OEIS entry A217596 (March 19, 2013), the expansion of $x/\mathrm{reversion}(x - x^2 - x^3)$, lists exactly this binomial sum, so that the right side of (1.1) is $-\mathrm{A217596}(j+1)$ [14]; in our notation its generating function is $1 - tS_*$, where $S_* = (1+\xi)/(1-\xi-\xi^2)$ for the series ξ in Theorem 1.1. The same day Paul D. Hanna added functional equations for this generating function A, the first being $A = 1 - t/A - t^2/A^2$; clearing denominators gives the cubic $A^3 = A^2 - tA - t^2$, which Burcroff, Lee, Mou, Musiker and Reineke used to note that the conjectured wall function is algebraic [5, §9.3], and which is equivalent to (1.2) under $A = 1 - tS$. According to a comment of Muhammed Sefa Saydam on the entry (January 27, 2025), $-\mathrm{A217596}(n)$ counts the dissections of a convex $(n+2)$-gon into triangles and quadrilaterals by noncrossing diagonals none of which passes through a chosen vertex; so $\tau^{1,5}(2j, j)$ is this count for a $(j+3)$-gon (for example 2 for a square and 5 for a pentagon). We do not use this interpretation.

Appendix C Further consequences

The coefficient formula has the following consequences.

Remark C.1 (Algebraicity). In particular, the wall function $S(x^2y^5)$ on the ray $(2,5)$ of $\mathfrak{D}(1+x, 1+y^5)$ is algebraic over $\mathbb{Q}(x^2y^5)$, by (1.2). Burcroff, Lee, Mou, Musiker and Reineke deduced this, and an exponential growth rate $27/5$ of the coefficients, from the conjectured formula [5, §9.3], in connection with the question, which they attribute to Kontsevich, whether every wall function of a cluster scattering diagram is algebraic. Theorem 1.1, together with the deduction in [5, §9.3], makes both statements unconditional for this ray.

Remark C.2 (Other exponents). Sections 4.1–6 do not use the value 5 except through the exponent of T. For an integer $c \geq 1$ the same argument applies to T^c and the ray $(2,1)$ of $\mathfrak{D}((1+t)^c, 1+t)$: it shows that this ray carries $\rho_c^{c/2}$, where $\rho_c \in \Lambda$ is the unique solution of

$$2\rho' - \rho + 1 - (c-2)(t\rho^{c/2-1})' - t\rho^{c/2-1} = 0.$$

This general form is a by-product of the written argument and is not part of the formalization, which treats $c = 5$ only; for $c = 3, 6, 7$ it agrees with the factorization of T^c through order t^5.

Appendix D Relation to the formal proof

The formal argument uses factorization and a coefficient recurrence in place of logarithms and Lagrange inversion.

The comparison of graded components in Proposition 5.6 is replaced by the uniqueness of factorizations in $(1 + \mathcal{O}_{\leq -1}) \cdot \mathbb{Q}[\![t]\!]^\times \cdot (1 + \mathcal{O}_{\geq 1})$ applied directly to (5.3). Lemma 6.1 is used in a form multiplied through by $\phi(u)\phi(w)/(uw)$. The formal argument starts with the binomial series rather than S_*. Its coefficients and those of the root of (1.2) satisfy the same second-order linear recurrence with the same initial values; for the sequence A217596, Peter Bala recorded this recurrence in 2015 [14]. The formal argument then derives (6.2) for the square and cube of that series from (1.2).

The formal definitions use the consistency condition of Definition 3.1, identified with [10, §1.3] by Remark B.1. The levelwise reading of the infinite product is equivalent to the $\mathfrak{m}$-adic one, and

the ray of slope $5/2$ is $(2,5)$ in Definition 3.3. The identification with the coefficients in [6] is the reading explained in Remark B.2.

Nearby work. Hipolito Treffinger [17] studies scattering diagrams for Artin algebras; nothing from that work is used here.

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