A four-dimensional 4-reptile simplex

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Abstract

A d-dimensional simplex is a k-reptile if it can be tiled by k mutually congruent simplices similar to it. Matoušek and Safernová conjectured that for $d \geq 3$ a k-reptile d-simplex exists only when $k = m^d$ for an integer m, and proved this for $d = 3$. Kynčl and Patáková proved that in dimension four k must be a square, noted that every k-reptile simplex known in dimension $d \geq 3$ has $k = m^d$, and asked whether a four-dimensional k-reptile simplex exists with k a square but not a fourth power. We show that the simplex with vertices $(0,0,0,0)$, $(1,0,0,0)$, $(\frac{1}{2},\frac{1}{2},0,0)$, $(\frac{2}{3},\frac{1}{3},\frac{1}{3},0)$ and $(\frac{3}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4})$, the fundamental alcove of the affine Weyl group of type F_4, is a 4-reptile: it is tiled by four congruent copies of itself scaled by $1/\sqrt{2}$, two of them mirror images. This answers the question of Kynčl and Patáková with $k = 4$ and disproves the conjecture of Matoušek and Safernová in dimension four. The subdivision itself is due to Felikson and Tumarkin, who found in 2004 that the affine reflection group of type F_4 has a reflection subgroup of index 4 whose fundamental simplex is similar to the alcove. What we add is the observation that this settles the reptile question, a short direct proof of the tiling, and exact computer checks of it.

This work has not yet been reviewed by experts. The proofs, the computer checks and the text were produced by AI models (from OpenAI and Anthropic), working under the author's direction.

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1 Introduction

1.1 Main result

A tiling of a closed set $X \subset \mathbb{R}^d$ is a decomposition of X into finitely many closed sets with nonempty, pairwise disjoint interiors. Following Kynčl and Patáková [4], a d-dimensional simplex S is a *k-reptile* if it can be tiled by k simplices that are all mutually congruent and similar to S; as they emphasize, "congruence includes mirror symmetries" [4, p. 1]. Throughout, let

$$S = \operatorname{conv}\{v_0, v_1, v_2, v_3, v_4\}, \qquad \text{where}$$
$$v_0 = (0,0,0,0), \quad v_1 = (1,0,0,0), \quad v_2 = (\tfrac{1}{2},\tfrac{1}{2},0,0), \quad v_3 = (\tfrac{2}{3},\tfrac{1}{3},\tfrac{1}{3},0), \quad v_4 = (\tfrac{3}{4},\tfrac{1}{4},\tfrac{1}{4},\tfrac{1}{4}).$$

Theorem 1. *The simplex $S \subset \mathbb{R}^4$ is a 4-reptile. More precisely, S is the union of four simplices $S_0, \dots, S_3$ with pairwise disjoint interiors, each the image of S under a similarity of ratio $1/\sqrt{2}$. Two of the four similarities preserve orientation and two reverse it.*

Since $4 = 2^2$ and 4 is not a fourth power, Theorem 1 answers the question of Kynčl and Patáková stated below, with $k = 4$, and gives a counterexample, in dimension $d = 4$, to the conjecture of Matoušek and Safernová.

The simplex S is the fundamental alcove of the affine Weyl group of type F_4 (Section 3). A linear map φ that is $\sqrt{2}$ times an isometry carries the hyperplane arrangement of this group into itself, so $\varphi(S)$ is a union of alcoves, 4 of them by volume. Section 2 proves Theorem 1 directly, by writing $\varphi(S)$ and its four pieces as solution sets of linear inequalities; the proof needs no Lie theory and no computer. Section 4 describes exact computer checks of the same tiling.

1.2 The question

Hertel [3, pp. 650–652] proved that the simplices he calls H-simplices, and more generally his H-simplotopes, are m^d-replicating (tiled by m^d congruent copies of themselves scaled down) for every integer $m \geq 2$. His Conjecture 2 [3, p. 652] asserts that, for $d > 2$, an f-replicating d-simplex must have $f = m^d$ for an integer $m \geq 2$ and must be an H-simplex. Theorem 1 also refutes this conjecture in dimension four, since 4 is not a fourth power.

Matoušek and Safernová [5, Conjecture 1.2] conjectured: "If $d \geq 3$ and a d-dimensional k-reptile simplex exists, then $k = m^d$ for some natural number $m \geq 2$." They proved it for $d = 3$; Kynčl and Patáková gave a simpler proof [4, Theorem 1.2] and proved that a four-dimensional k-reptile simplex can exist only when k is the square of a positive integer [4, Theorem 1.3]. In their abstract they note that "the only k-reptile simplices that are known for $d \geqslant 3$, have $k = m^d$". They then ask [4, Problem 1.4; Problem 2 in the arXiv version] whether there is a four-dimensional k-reptile simplex with k the square of a non-square integer, that is, with k a square but not a fourth power. Theorem 1 answers this with $k = 4$, the square of 2. It is consistent with [4, Theorem 1.3], since $k = 4$ is a square, and it leaves the case $d = 3$ untouched.

1.3 Credit

The tiling is not new as geometry. Felikson and Tumarkin [2, Lemma 10] consider Coxeter simplices $F \neq P$ in $\mathbb{E}^n$ with the same Coxeter diagram, $\Sigma(F) = \Sigma(P)$, and with $G_P \subset G_F$ for the groups generated by the reflections in their facets. They prove that $1 < [G_F : G_P] < 2^n$ holds exactly in three cases, the last being "$\Sigma(F) = \Sigma(P) = \widetilde{F}_4$, $[G_F : G_P] = 4$". Equal Coxeter diagrams make P similar to F. A fundamental simplex of an index-4 reflection subgroup of G_F is a union of 4 fundamental simplices of G_F, which are congruent. So their $\widetilde{F}_4$ case is a simplex similar to the F_4 alcove and tiled by four alcoves: in substance, Theorem 1. Their paper does not discuss reptiles, and we found no reference connecting it to the reptile question. Our contribution is that connection, the explicit tiling with a direct proof, and its checks. The other two cases of their lemma, of types $\widetilde{C}_2$ and $\widetilde{G}_2$, give the classical 2-reptile isosceles right triangle and 3-reptile 30-60-90 triangle.

2 Proof of Theorem 1

We describe S by inequalities. For $x = (x_1, x_2, x_3, x_4)$ let

$$\ell_0(x) = 1 - x_1 - x_2, \quad \ell_1(x) = x_2 - x_3, \quad \ell_2(x) = x_3 - x_4, \quad \ell_3(x) = x_4, \quad \ell_4(x) = x_1 - x_2 - x_3 - x_4.$$

Lemma 2. *$S = \{x \in \mathbb{R}^4 : \ \ell_i(x) \geq 0$ for $i = 0, \ldots, 4\}$, that is, $S = \{x : \ x_2 \geq x_3 \geq x_4 \geq 0, \ x_1 \geq x_2 + x_3 + x_4, \ x_1 + x_2 \leq 1\}$.*

Proof. Each vertex of S makes exactly one of the five functions nonzero, and that one positive: $\ell_0(v_0) = 1$, $\ell_4(v_1) = 1$, $\ell_1(v_2) = \frac{1}{2}$, $\ell_2(v_3) = \frac{1}{3}$ and $\ell_3(v_4) = \frac{1}{4}$, and every other value $\ell_i(v_j)$ is 0. The points v_j are affinely independent, since $v_1, \dots, v_4$ are linearly independent. Let $\lambda_0, \dots, \lambda_4$ be the barycentric coordinates with respect to $v_0, \dots, v_4$. For each vertex there is one function, say $\ell_{\sigma(j)}$ for v_j, that is nonzero on it; the affine functions $\ell_{\sigma(j)}/\ell_{\sigma(j)}(v_j)$ and λ_j agree on all five vertices, hence everywhere. So $S = \{\lambda_j \geq 0 \ \forall j\} = \{\ell_i \geq 0 \ \forall i\}$. □

Let $\varphi(x) = (x_1 + x_2,\ x_1 - x_2,\ x_3 + x_4,\ x_3 - x_4)$. Its matrix Φ satisfies $\Phi^{\mathsf{T}}\Phi = 2I$, so $\varphi/\sqrt{2}$ is an orthogonal map and $T = \varphi(S)$ is similar to S with ratio $\sqrt{2}$; also $\det \Phi = 4$. Substituting $x_1 = \frac{y_1+y_2}{2}$, $x_2 = \frac{y_1-y_2}{2}$, $x_3 = \frac{y_3+y_4}{2}$, $x_4 = \frac{y_3-y_4}{2}$ into Lemma 2 gives

$$T = \{y \in \mathbb{R}^4 :\ y_1 \leq 1,\ y_1 \geq y_2 + y_3 + y_4,\ y_2 \geq y_3 \geq y_4 \geq 0\}. \tag{1}$$

Its vertices are $\varphi(v_0) = (0,0,0,0)$, $\varphi(v_1) = (1,1,0,0)$, $\varphi(v_2) = (1,0,0,0)$, $\varphi(v_3) = (1, \frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ and $\varphi(v_4) = (1, \frac{1}{2}, \frac{1}{2}, 0)$. Note that $S \subset T$ by Lemma 2 and (1).

Define four isometries of $\mathbb{R}^4$:

$$g_0(x) = x, \qquad g_1(x) = (1 - x_2,\ 1 - x_1,\ x_3,\ x_4),$$
$$g_2(x) = (1 - x_3,\ 1 - x_1,\ x_2,\ x_4), \qquad g_3(x) = (1 - x_4,\ 1 - x_1,\ x_2,\ x_3).$$

Each is a signed permutation of coordinates followed by a translation. The linear parts of g_0, g_1, g_2, g_3 have determinants $1, -1, 1, -1$.

Lemma 3. *Let $P_i = g_i(S)$. Then*

$$P_0 = T \cap \{y :\ y_1 + y_2 \leq 1\}, \qquad P_1 = T \cap \{y :\ y_1 + y_3 \leq 1 \leq y_1 + y_2\},$$
$$P_2 = T \cap \{y :\ y_1 + y_4 \leq 1 \leq y_1 + y_3\}, \qquad P_3 = T \cap \{y :\ 1 \leq y_1 + y_4\}.$$

Proof. Since g_i is a bijection, $y \in P_i$ exactly when $x = g_i^{-1}(y)$ satisfies the five inequalities of Lemma 2. We write them out in y for each i.

For $i = 0$: $y_2 \geq y_3 \geq y_4 \geq 0$, $y_1 \geq y_2 + y_3 + y_4$ and $y_1 + y_2 \leq 1$. The remaining inequality $y_1 \leq 1$ of (1) follows from $y_1 \leq 1 - y_2 \leq 1$.

For $i = 1$, put $x = (1 - y_2,\ 1 - y_1,\ y_3,\ y_4)$. The five inequalities become $y_1 + y_3 \leq 1$, $y_3 \geq y_4$, $y_4 \geq 0$, $y_1 \geq y_2 + y_3 + y_4$ and $y_1 + y_2 \geq 1$. The inequalities $y_1 \leq 1$ and $y_2 \geq y_3$ of (1) follow: $y_1 \leq 1 - y_3 \leq 1$, and $y_1 + y_2 \geq 1 \geq y_1 + y_3$.

For $i = 2$, put $x = (1 - y_2,\ y_3,\ 1 - y_1,\ y_4)$. The inequalities become $y_1 + y_3 \geq 1$, $y_1 + y_4 \leq 1$, $y_4 \geq 0$, $y_1 \geq y_2 + y_3 + y_4$ and $y_2 \geq y_3$. The inequalities $y_3 \geq y_4$ and $y_1 \leq 1$ follow from $y_1 + y_3 \geq 1 \geq y_1 + y_4$ and $y_1 \leq 1 - y_4 \leq 1$.

For $i = 3$, put $x = (1 - y_2,\ y_3,\ y_4,\ 1 - y_1)$. The inequalities become $y_3 \geq y_4$, $y_1 + y_4 \geq 1$, $y_1 \leq 1$, $y_1 \geq y_2 + y_3 + y_4$ and $y_2 \geq y_3$. The inequality $y_4 \geq 0$ follows from $y_4 \geq 1 - y_1 \geq 0$.

In each case the inequalities obtained, together with the ones shown to follow, are exactly those of (1) and of the stated region. □

Proof of Theorem 1. On T we have $y_2 \geq y_3 \geq y_4$, so $y_1 + y_2 \geq y_1 + y_3 \geq y_1 + y_4$. For $y \in T$, the number 1 lies above $y_1 + y_2$, between two consecutive terms of this chain, or below $y_1 + y_4$, so y lies in at least one of the four regions of Lemma 3. Hence $T = P_0 \cup P_1 \cup P_2 \cup P_3$. Within T, two different regions intersect only inside one of the hyperplanes $y_1 + y_j = 1$ $(j = 2, 3, 4)$, so the P_i have pairwise disjoint interiors. Each $P_i = g_i(S)$ is congruent to S.

Apply φ^{-1}. The sets $S_i = \varphi^{-1}(P_i)$ tile $\varphi^{-1}(T) = S$, and S_i is the image of S under the similarity $\varphi^{-1} \circ g_i$ of ratio $1/\sqrt{2}$. These four similarities differ from one another by isometries, so the S_i are mutually congruent. Since $\det \Phi > 0$, the similarity $\varphi^{-1} \circ g_i$ preserves orientation exactly when g_i does, that is, for $i = 0, 2$. □

The tiling is a chain: consecutive pieces share a facet. The vertices of the pieces are listed in Appendix A.

Remark 4. The mirror images cannot be avoided by relabeling: S has no nontrivial symmetry, since its ten edge lengths admit no nonidentity permutation of the vertices preserving them. Whether some other tiling of S by four copies uses only directly congruent ones is not settled here.

3 Where the tiling comes from

Take the root system of type F_4 in Bourbaki's coordinates [1, Plate VIII]: its 48 roots are $\pm e_i$, $\pm e_i \pm e_j$ $(i < j)$ and $\frac{1}{2}(\pm e_1 \pm e_2 \pm e_3 \pm e_4)$, with simple roots $e_2 - e_3$, $e_3 - e_4$, e_4, $\frac{1}{2}(e_1 - e_2 - e_3 - e_4)$ and highest root $e_1 + e_2$. The fundamental alcove $\{x : \langle \alpha, x\rangle \geq 0$ for the simple roots, $\langle e_1 + e_2, x\rangle \leq 1\}$ is exactly the description of S in Lemma 2. The affine Weyl group is generated by the reflections in the hyperplanes $\langle \alpha, x\rangle = n$ (α a root, $n \in \mathbb{Z}$), and its alcoves, the closures of the regions of this arrangement, are congruent copies of S tiling $\mathbb{R}^4$. The matrix Φ is symmetric with $\Phi^2 = 2I$, so $\varphi^{-1}(y) = \Phi y/2$ and φ maps the hyperplane $\langle \alpha, x\rangle = n$ onto $\langle \Phi\alpha, y\rangle = 2n$. A direct check of the 48 roots shows that Φ maps each short root ($\pm e_i$ and $\frac{1}{2}(\pm e_1 \pm e_2 \pm e_3 \pm e_4)$) to a long root and each long root to twice a short root. So the image hyperplane is $\langle \beta, y\rangle = 2n$ or $\langle \beta, y\rangle = n$ for a root β, again a hyperplane of the arrangement. Hence $T = \varphi(S)$, the image of a region of the arrangement, is a union of alcoves, $\det \Phi = 4$ of them by volume. These are the pieces P_i, and the third program in Section 4 confirms that each is an alcove. This is the $\widetilde{F}_4$ case of [2, Lemma 10], written in coordinates.

Corollary 5. *For all integers $a \geq 0$ and $m \geq 1$, S is a $4^a m^4$-reptile.*

Proof. Iterating Theorem 1 a times tiles S by 4^a congruent copies scaled by $2^{-a/2}$. The dilate mS is the set $\{x : \langle \alpha, x\rangle \geq 0$ for the simple roots, $\langle e_1 + e_2, x\rangle \leq m\}$, whose facets lie in hyperplanes of the arrangement, so it is a union of alcoves, m^4 of them by volume. Scaling by $1/m$ tiles S by m^4 copies of S/m; apply this inside each of the 4^a pieces. □

For a odd, $4^a m^4$ is a square but not a fourth power. Whether $k = 9, 25, 36, \ldots$ ($m = 3, 5, 6, \ldots$ in the question of Kynčl and Patáková) occur in dimension four, and whether any k-reptile simplex with $k \neq m^d$ exists for $d \geq 5$, remain open.

4 Computational verification

The proof above is complete without a computer. The same tiling is also checked in exact rational arithmetic by three programs, which read a certificate file listing S, Φ and, for each piece, an orthogonal matrix M_i, a translation t_i (so that $P_i = M_i S + t_i$) and the five vertices, together with one separating hyperplane for each pair of pieces.

1. The first program checks that S is nondegenerate, $\Phi^{\mathsf{T}}\Phi = 2I$, each M_i is orthogonal and the listed vertices are $M_i S + t_i$, every piece vertex has nonnegative barycentric coordinates in T, each pair of pieces lies on opposite closed sides of its listed hyperplane, and the four volumes add up to $\operatorname{vol} T = \frac{1}{144} = 4 \operatorname{vol} S$.

2. The second, written separately from the first, compares S and Φ with the statement above, ignores the listed hyperplanes, checks congruence by the ten squared edge lengths as well as by M_i, checks containment by the facet inequalities of T, and shows that each pairwise intersection has dimension at most 3 by enumerating its vertices exactly; then it checks the volume sum.

3. The third evaluates all 48 roots of F_4 on the vertices and checks that S and each piece lie in a closed alcove, that the pieces are distinct, have the volume of S and lie in T, and that $\operatorname{vol} T = 4 \operatorname{vol} S$. As S is an alcove, the pieces are then four distinct alcoves in T.

Closed convex pieces inside T with disjoint interiors and total volume $\operatorname{vol} T$ cover T, so each of the first two programs alone verifies the tiling of T and hence Theorem 1. Six corrupted certificates (a moved vertex, a map that is not a similarity, a duplicated piece, a dropped piece, a shifted separating hyperplane, and a non-orthogonal M_i) are each rejected by every program that reads the corrupted field. The whole check takes under a second. The written proof of Section 2 does not depend on these programs; they check the same four pieces. The certificate, the programs and the command that runs them are in the folder accompanying this paper, whose README describes them.

Appendix A The vertices of the pieces

With v_2, v_3, v_4 as in the introduction and $w_1 = (1,1,0,0)$, $w_2 = (1, \frac{1}{2}, \frac{1}{2}, 0)$, $w_3 = (1, \frac{1}{3}, \frac{1}{3}, \frac{1}{3})$, the pieces $P_i = g_i(S)$ of T are

$$P_0 = \operatorname{conv}\{0,\, v_1,\, v_2,\, v_3,\, v_4\} = S, \qquad P_1 = \operatorname{conv}\{w_1,\, v_1,\, v_2,\, v_3,\, v_4\},$$
$$P_2 = \operatorname{conv}\{w_1,\, v_1,\, w_2,\, v_3,\, v_4\}, \qquad P_3 = \operatorname{conv}\{w_1,\, v_1,\, w_2,\, w_3,\, v_4\}.$$

Each piece arises from the previous one by replacing one vertex, and $T = \operatorname{conv}\{0, v_1, w_1, w_2, w_3\}$. The points v_2, v_3, v_4 lie on the edges of T from 0: $v_2 = \frac{1}{2}w_1$, $v_3 = \frac{2}{3}w_2$, $v_4 = \frac{3}{4}w_3$. The tiling of S in Theorem 1 is the image of this one under φ^{-1}.

References

[1] N. Bourbaki, *Lie Groups and Lie Algebras, Chapters 4–6*, Springer, 2002, Plate VIII (system of type F_4).

[2] A. A. Felikson and P. V. Tumarkin, Reflection subgroups of Euclidean reflection groups, *Sb. Math.* **196** (2005), 1349–1369; arXiv:math/0402403 (February 25, 2004), Lemma 10.

[3] E. Hertel, A class of self-similar d-polytopes, *Rev. Roumaine Math. Pures Appl.* **50** (2005), 647–655.

[4] J. Kynčl and Z. Patáková, On the nonexistence of k-reptile simplices in $\mathbb{R}^3$ and $\mathbb{R}^4$, *Electron. J. Combin.* **24**(3) (2017), #P3.1, doi:10.37236/6113; arXiv:1602.04668.

[5] J. Matoušek and Z. Safernová, On the nonexistence of k-reptile tetrahedra, *Discrete Comput. Geom.* **46** (2011), 599–609, doi:10.1007/s00454-011-9334-z; arXiv:1006.1807.