Combinatorial periods of asymmetric billiard trajectories in the regular pentagon

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Abstract

We prove that every even integer other than 2, 12, 14 and 18 is the combinatorial period, the number of bounces before repeating, of a periodic billiard trajectory in the regular pentagon that does not have five-fold rotational symmetry. Davis and Lelièvre conjectured this in 2018 and checked it up to period 4000, and Kontorovich and Zhang proved it for almost every even number. As they explain, a theorem of Davis, Fuchs and Tabachnikov turns the question into one about vectors. Each vector $(a+b\varphi, c+d\varphi)$ obtained from $(0,1)$ by products of four explicit 2×2 matrices, where φ is the golden ratio and a, b, c, d are integers, has the half-period $\ell = a+b+c+d$, and when a congruence modulo 5 holds it gives such a trajectory of period $n = 2\ell$. Varying the power of one of the two triangular matrices in such a product gives half-periods in an arithmetic progression. We give 379 of these progressions, which together contain every half-period $\ell \geq 2$ except nine values. Six of the nine come from explicit trajectories checked in exact arithmetic; the other three are halves of excluded periods. The covering argument is checked in Lean, assuming the cited theorem. Whether the four excluded periods occur is not treated.

This work has not yet been reviewed by experts. The proofs, the Lean formalization and the text were produced by AI models (from OpenAI and Anthropic), working under the author's direction.

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MSC 2020: 37C83, 37E35, 11B25.

1 Introduction

1.1 Main result

A billiard trajectory in a polygon moves in straight lines and reflects off the sides with equal angles of incidence and reflection; it is *nonsingular* if it never hits a vertex. A nonsingular trajectory is *periodic* if it returns to its initial point with its initial direction, and its *combinatorial period* is the number of reflections in a least period, the "number of edge crossings before repeating" in the words of Davis and Lelièvre [2, §1]. Kontorovich and Zhang call a trajectory *symmetric* when it has 5-fold rotational symmetry [3, §1]: the rotation R by $2\pi/5$ about the center maps it to itself. Otherwise it is *asymmetric*. Whether trajectories are taken with or without orientation does not matter here: if R mapped a trajectory onto itself with its orientation reversed, then so would R^5, the identity, since 5 is odd.

Theorem 1.1. *For every even integer $n > 0$ with $n \notin \{2, 12, 14, 18\}$, there is a nonsingular periodic billiard trajectory in the regular pentagon whose combinatorial period is n and which does not have 5-fold rotational symmetry.*

Whether the four excluded periods occur for asymmetric trajectories is a separate question, which Theorem 1.1 does not address.

1.2 Background

Davis and Lelièvre showed that every positive multiple of 10 is the combinatorial period of a rotationally symmetric trajectory [2, Proposition 4.2], and wrote [2, §4.1]:

> Based on computer evidence, we conjecture the following: Conjecture 4.3. Every even number arises as the combinatorial period of an asymmetric periodic billiard trajectory on the pentagon, other than 2, 12, 14 and 18. We have verified this up to period 4000.

Theorem 1.1 is the existence statement of this conjecture. Kontorovich and Zhang restated it as their Conjecture 1.1 [3, §1]. They proved [3, Theorem 1.2] that "Conjecture 1.1 is true in density, that is, almost every even number arises", with at most $\ll X/(\log X)^{1/40}$ exceptions up to X, and showed that evenness is the only congruence obstruction [3, Theorem 1.4]. They explained why the orbital circle method does not apply here [3, Remark 1.5], and that their sieve method misses $\gg X/\log X$ values [3, Remark 4.5].

1.3 Proof overview

Section 2 states the theorem of Davis, Fuchs and Tabachnikov [1, Theorem 11] that we use, in the form of [3, Theorem 2.1]. To each vector $v = (a + b\varphi, c + d\varphi)$ of an orbit Λ of a semigroup Γ^+ of 2×2 matrices it attaches the half-period $\ell(v) = a+b+c+d$, and when a congruence modulo 5 holds it gives an asymmetric trajectory of period $n = 2\ell(v)$. So Theorem 1.1 asks for every half-period $\ell = n/2$ with $\ell \notin \{1, 6, 7, 9\}$. Section 3 shows that one-parameter families in Γ^+ built on a parabolic generator give arithmetic progressions of half-periods, and Section 4 gives 379 progressions that together contain every $\ell \geq 2$ outside the set

$$E = \{4, 5, 6, 7, 9, 15, 21, 29, 51\}.$$

Section 5 treats $\ell \in \{4, 5, 15, 21, 29, 51\}$, the periods 8, 10, 30, 42, 58 and 102, by six explicit trajectories, checked in exact arithmetic without the cited theorem. The remaining values $\ell \in \{1, 6, 7, 9\}$ are the half-periods of the excluded periods. The new step is the covering: each family gives all large members of one residue class, so finitely many families whose classes exhaust the residues modulo a fixed number, with the few small values checked separately, give every half-period at once. The arithmetic of Sections 3 and 4 is checked in Lean (Section 6).

2 The cited theorem

Let $\varphi = (1 + \sqrt{5})/2$, and let Γ^+ be the semigroup generated by

$$A = \begin{pmatrix} 1 & \varphi \\ 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 \\ \varphi & 1 \end{pmatrix}, \quad C = \begin{pmatrix} \varphi & \varphi \\ 1 & \varphi \end{pmatrix}, \quad D = \begin{pmatrix} \varphi & 1 \\ \varphi & \varphi \end{pmatrix},$$

as in [3, §2]. All entries of the generators lie in $\mathbb{Z}_{\geq 0}[\varphi] = \{p + q\varphi : p, q \in \mathbb{Z}_{\geq 0}\}$, so every vector of the orbit $\Lambda = \Gamma^+ \cdot (0,1)^t$ has the form $v = (a + b\varphi,\ c + d\varphi)$ with $a, b, c, d \geq 0$ integers, uniquely determined since φ is irrational. Put

$$\ell(v) = a + b + c + d, \qquad \lambda(v) = (d - b) + 2(c - a).$$

Both are $\mathbb{Z}$-linear in (a, b, c, d); we call $\ell(v)$ the *half-period* of v.

The geometry behind the cited theorem is as follows; the proof uses only its consequence, Corollary 2.2. Billiard trajectories in the pentagon correspond to straight lines on the double pentagon, a translation surface made of two copies of the pentagon, by the usual unfolding. Cutting and translating pieces of the double pentagon and applying a linear map, operations that preserve straight lines, gives another translation surface, the golden L [3, §2, Figure 2]. A *saddle connection* on such a surface is a straight segment that starts and ends at singular points, which come from the vertices of the pentagons, and passes through none in between; it is recorded by its displacement vector in the plane. A direction is *periodic* when the straight lines in it are closed, apart from those that meet a singular point. In each periodic direction, Kontorovich and Zhang name one saddle connection short and one long, and all the proof needs is that every vector of Λ is a long one. They write [3, §2]: "Each periodic direction in the golden L has a short and long saddle connection, and all long saddle connections with directions in the first quadrant are given by $\Lambda_L = \Gamma^+ \cdot (0,1)^t$." They then state:

Theorem 2.1 ([3, Theorem 2.1], citing [1, Theorem 11])**.** *"Given a long saddle connection $(a + b\varphi, c + d\varphi) \in \Gamma^+ \cdot (0,1)^t$ in the golden L, let $\ell = a + b + c + d$. The combinatorial period of the corresponding pentagon billiard trajectory is given either by $L = 2\ell$ or $L = 10\ell$, depending on whether $(d - b) + 2(c - a) \equiv 0 \pmod 5$ or not (resp.), and corresponding to whether the associated trajectory is asymmetric or not."*

By the sentence before it, the hypothesis of Theorem 2.1 holds for every $v \in \Lambda$. We use only its first case.

Corollary 2.2. *If $v \in \Lambda$ and $\lambda(v) \equiv 0 \pmod 5$, then $2\ell(v)$ is the combinatorial period of an asymmetric periodic billiard trajectory in the regular pentagon.*

Theorem 1.1 therefore follows from Proposition 4.1 below, on values of ℓ, together with Proposition 5.1 for six small periods.

3 Parabolic families

Write $v \in \Lambda$ as the column $(a, b, c, d) \in \mathbb{Z}^4_{\geq 0}$. Using $\varphi^2 = \varphi + 1$, each generator acts on these coordinates by a nonnegative integer matrix; for example A sends (a, b, c, d) to $(a+d,\ b+c+d,\ c,\ d)$. Two of the generators are parabolic:

$$A = I + N_A, \quad N_A = \begin{pmatrix} 0 & \varphi \\ 0 & 0 \end{pmatrix}, \qquad B = I + N_B, \quad N_B = \begin{pmatrix} 0 & 0 \\ \varphi & 0 \end{pmatrix}, \qquad N_A^2 = N_B^2 = 0,$$

so $P^n = I + nN_P$ for $P \in \{A, B\}$ and all $n \geq 0$.

Lemma 3.1. *Let u, w be words in A, B, C, D, let $P \in \{A, B\}$, and put $v_n = uP^n w \cdot (0,1)^t \in \Lambda$. Then $\ell(v_n) = \ell_0 + n\alpha$ and $\lambda(v_n) = \lambda_0 + n\beta$ for integers $\ell_0, \alpha, \lambda_0, \beta$, where $\alpha = \ell(uN_P w \cdot (0,1)^t) \geq 0$. If $\alpha > 0$ and $\beta \not\equiv 0 \pmod 5$, let $n_0 \in \{0, \ldots, 4\}$ satisfy $\lambda_0 + n_0\beta \equiv 0 \pmod 5$; then every $s + jm$*

with $j \geq 0$, where $s = \ell_0 + n_0\alpha$ and $m = 5\alpha$, is $\ell(v)$ for some $v \in \Lambda$ with $\lambda(v) \equiv 0 \pmod 5$. If $\alpha > 0$, $\beta \equiv 0$ and $\lambda_0 + n_0\beta \equiv 0 \pmod 5$ for some $n_0 \geq 0$, the same holds with $s = \ell_0 + n_0\alpha$ and $m = \alpha$.

Proof. Since u acts linearly, $v_n = uw \cdot (0,1)^t + n\, uN_P w \cdot (0,1)^t$, and ℓ and λ are linear. The orbit vectors v_{n_0+5j}, or v_{n_0+j} when $\beta \equiv 0$, have $\lambda \equiv 0 \pmod 5$ and $\ell = s + jm$. □

Example 3.2. Take u and w empty and $P = A$. Then $A^n(0,1)^t = (n\varphi, 1)^t$, so $(a,b,c,d) = (0,n,1,0)$, $\ell = n+1$ and $\lambda = 2-n$. With $n = 2+5j$ we get asymmetric vectors with $\ell = 3+5j$, so every 2ℓ with $\ell \equiv 3 \pmod 5$ and $\ell \geq 3$ is an asymmetric period.

A single family covers one residue class modulo its m, from its start s on. Kontorovich and Zhang's proof of their density theorem also produces periods from products of parabolic powers, the vectors obtained from $B^m A^n B^k$ [3, (4.1)], with exponents restricted to progressions modulo 5 so that the trajectories stay asymmetric [3, §4]; there ℓ is cubic in the exponents, and a sieve gives almost all values. Here only one exponent varies, so ℓ is linear in it, and the question becomes one of covering residue classes.

4 The covering

Proposition 4.1. *For every integer $\ell \geq 2$ with $\ell \notin E$, there is $v \in \Lambda$ with $\ell(v) = \ell$ and $\lambda(v) \equiv 0 \pmod 5$.*

The proof is a finite verification organized as a tree. Let

$$M = 154\,954\,800 = 2^4 \cdot 3^2 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13 \cdot 43,$$

and fix the order of its prime factors $(p_1, \dots, p_{12}) = (5,5,2,3,2,7,2,3,2,11,13,43)$. A *node* is a pair (Q, r) with $Q = p_1 \cdots p_k$ for some $0 \leq k \leq 12$ and $0 \leq r < Q$; it stands for the residue class r mod Q. The children of (Q, r) are the p_{k+1} nodes $(Qp_{k+1},\ r + Qt)$ with $0 \leq t < p_{k+1}$. Let $E' = E \cup \{0, 1\}$. A family (m, s) from Lemma 3.1 *closes* the node (Q, r) if

$$m \mid Q, \qquad r \equiv s \pmod m, \qquad \text{and} \qquad s \leq r \ \text{ or } \ (r \in E' \text{ and } s \leq r + Q).$$

Lemma 4.2. *If (m, s) closes (Q, r), then every integer $x \geq 0$ with $x \equiv r \pmod Q$ and $x \notin E'$ satisfies $x \equiv s \pmod m$ and $x \geq s$.*

Proof. Since $m \mid Q$, $x \equiv r \equiv s \pmod m$. As $0 \leq r < Q$ and $x \equiv r$, either $x = r$ or $x \geq r + Q$. If $s \leq r$ then $x \geq s$. If $r \in E'$, then $x \neq r$ because $x \notin E'$, so $x \geq r + Q \geq s$. □

Proof of Proposition 4.1. We give 379 families, each given by words u, w, a parabolic generator P and an exponent n_0, with m up to 17 325, s up to 16 221, and words w of length up to 44 and u up to 7; they are listed in the Lean source (Section 6). For each family the hypotheses of Lemma 3.1 hold, with the stated m and s. Starting from the root $(1, 0)$, we expand a node into its children until some family closes it. The resulting tree has 2 825 nodes, and every leaf is closed by one of the families; since $M = p_1 \cdots p_{12}$, every branch ends by depth 12. Given $\ell \geq 2$ with $\ell \notin E$, follow from the root the children containing ℓ, the class of ℓ modulo Q at each depth, to a leaf (Q, r). By Lemma 4.2, $\ell \equiv s \pmod m$ and $\ell \geq s$ for the family (m, s) closing it, so $\ell = s + jm$ with $j \geq 0$, and Lemma 3.1 gives v. □

The leaf condition lets the same tree cover the small values: a residue class whose least member lies in E' needs a family starting by its second member only. The families were found by a computer search over short words and a greedy choice of residue classes modulo M.

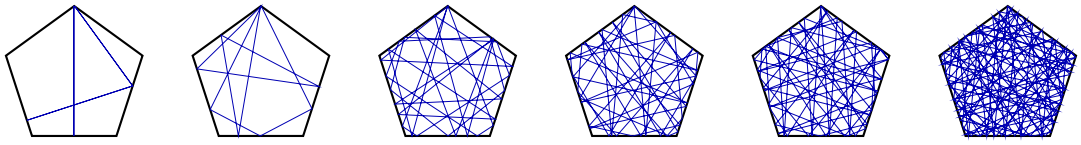


Figure 1: The six trajectories of Proposition 5.1, of periods 8, 10, 30, 42, 58 and 102, each starting on a side at distance 1/200 of the side's length from a vertex.

5 Six explicit trajectories

Place the regular pentagon with vertices $V_k = (\cos 72k^\circ, \sin 72k^\circ)$, $k = 0, \dots, 4$, and call the segment from V_k to V_{k+1} side k (indices mod 5). Write $\sigma = \sin 72^\circ$.

Proposition 5.1. *For each row of Table 1, the billiard trajectory starting at the point* $V_0 + \frac{1}{200}(V_1 - V_0)$ *of side* 0 *with direction* $(X, \sigma Y)$ *is nonsingular and periodic, has the combinatorial period shown, and does not have* 5*-fold rotational symmetry.*

period	X	Y
8	$-\frac{5}{2} - \frac{3}{2}\sqrt{5}$	0
10	$-\frac{15}{4} - \frac{7}{4}\sqrt{5}$	$\frac{5}{2} - \frac{1}{2}\sqrt{5}$
30	$-10 - \frac{11}{2}\sqrt{5}$	$\frac{15}{2} - \frac{1}{2}\sqrt{5}$
42	$-\frac{65}{4} - \frac{29}{4}\sqrt{5}$	$\frac{5}{2} - \frac{5}{2}\sqrt{5}$
58	$-\frac{85}{4} - \frac{41}{4}\sqrt{5}$	$-\frac{5}{2} + \frac{5}{2}\sqrt{5}$
102	$-\frac{155}{4} - \frac{71}{4}\sqrt{5}$	$-\frac{15}{2} - \frac{1}{2}\sqrt{5}$

Table 1: Initial directions $(X, \sigma Y)$ of the six trajectories.

Proof. All points and directions of these trajectories have coordinates $(x, \sigma y)$ with $x, y \in \mathbb{Q}(\sqrt{5})$, so each trajectory can be followed in exact arithmetic. At each step, the next side hit is the one at the least positive time among the sides other than the current one, the hit point is checked not to be a vertex, and the direction is reflected in that side. After n reflections, where n is the period in the table, the point, the side and the direction are again the initial ones, and not before. Rotation by 72° maps the initial point and direction to a pair that does not occur among the n states of the trajectory, so the rotation does not map the trajectory to itself. These finite computations were done by two independently written programs, one in the coordinates above and one in the cyclotomic field $\mathbb{Q}(\zeta_5)$ with the vertices at ζ_5^k (Section 6). □

Figure 1 shows the six trajectories. They were found in directions of small vectors of Λ, but Proposition 5.1 is checked directly, without Theorem 2.1.

Proof of Theorem 1.1. Let $n > 0$ be even with $n \notin \{2, 12, 14, 18\}$, and let $\ell = n/2$, so $\ell \geq 1$ and $\ell \notin \{1, 6, 7, 9\}$. If $\ell \in \{4, 5, 15, 21, 29, 51\}$, Proposition 5.1 gives the trajectory. Otherwise $\ell \geq 2$ and $\ell \notin E$, and Proposition 4.1 gives $v \in \Lambda$ with $\ell(v) = \ell$ and $\lambda(v) \equiv 0 \pmod 5$; by Corollary 2.2, $n = 2\ell(v)$ is the combinatorial period of an asymmetric periodic trajectory. □

6 Verification

The deduction of Theorem 1.1 from Theorem 2.1 and Proposition 5.1 is checked in Lean 4 [4], without external libraries. The Lean statement defines $\mathbb{Z}_{\geq 0}[\varphi]$, the generators, the orbit Λ as an inductive predicate, ℓ and the condition $\lambda \equiv 0 \pmod 5$. Billiards are not formalized: "some asymmetric periodic trajectory has combinatorial period n" is an opaque predicate AsymPeriod(n), which enters only through Theorem 2.1, stated as the axiom

$$\forall v,\ v \in \Lambda \ \to\ \lambda(v) \equiv 0 \pmod 5 \ \to\ \mathrm{AsymPeriod}(2\ell(v)),$$

and through the hypothesis that the six periods of Proposition 5.1 occur. The main formal theorem, `dl_conjecture_4_3_existence`, says: if AsymPeriod(n) holds for $n = 8, 10, 30, 42, 58, 102$, then it holds for every even $n > 0$ other than $2, 12, 14, 18$. With the axiom read as Corollary 2.2 and the hypothesis as Proposition 5.1, this is Theorem 1.1. Its proof uses the axiom and Lean's standard axioms only. Proposition 4.1 is the formal theorem `long_half_periods`, proved with no axiom beyond Lean's standard ones: Lemma 3.1 and Lemma 4.2 are proved in general, and the 379 families and the tree of Section 4 are checked by the kernel's evaluation of a decision procedure. Proposition 5.1 is checked outside Lean by the two exact programs, which also reject altered certificates. Appendix A gives the formal statements, and the accompanying README gives the commands to reproduce both checks.

Acknowledgments

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References

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Appendix A The formal statements

The definitions are in `Pentagon/Statement.lean`. An element $p + q\varphi$ of $\mathbb{Z}_{\geq 0}[\varphi]$ is a pair of natural numbers with the product $(p + q\varphi)(r + s\varphi) = pr + qs + (ps + qr + qs)\varphi$; a vector is a pair of these and a matrix a quadruple. The generators are `A`, `B`, `C`, `D`, written entrywise as in Section 2. `Orbit`

`v` is the least predicate containing $(0,1)$ and closed under the four generators; `ell v` is $a+b+c+d$ and `Asym v` is $((d-b)+2(c-a)) \bmod 5 = 0$ in the integers. The cited theorem is the axiom

```
axiom kz_thm_2_1 : ∀ v : Vec, Orbit v → Asym v → AsymPeriod (2 * ell v)
```

whose documentation quotes Theorem 2.1 and the sentence before it. The main theorem, in `Pentagon/Proof.lean`, is

```
theorem dl_conjecture_4_3_existence
    (small : AsymPeriod 8 ∧ AsymPeriod 10 ∧ AsymPeriod 30 ∧
      AsymPeriod 42 ∧ AsymPeriod 58 ∧ AsymPeriod 102) :
    ∀ n : Nat, 0 < n → n % 2 = 0 → n ≠ 2 → n ≠ 12 → n ≠ 14 → n ≠ 18 →
      AsymPeriod n
```

and Proposition 4.1 is

```
theorem long_half_periods (l : Nat) (h2 : 2 ≤ l) (hl : l ∉ exceptions) :
    ∃ v, Orbit v ∧ ell v = l ∧ Asym v
```

with `exceptions` the list $4, 5, 6, 7, 9, 15, 21, 29, 51$. The file `Pentagon/Cert.lean` lists the 379 families by the depth of the nodes they close, each as (m, s, w, P, u, n_0), and the prime order. `famOK` checks a family against Lemma 3.1 by computing $uw \cdot (0,1)^t$ and $uN_Pw \cdot (0,1)^t$, and `chk` walks the tree, expanding a node until `leafOK`, the closing condition, holds; both are evaluated by the kernel (`decide +kernel`), and their soundness lemmas `fam_sound`, `leaf_sound` and `chk_sound` are proved. The build file `PublicChecks.lean` restates these theorems with their full types and checks their axioms: `long_half_periods` uses only the standard axioms of Lean's logic, and `dl_conjecture_4_3_existence` uses these and the cited axiom. The development also proves `exceptions_combinatorial`: no $v \in \Lambda$ with $\lambda(v) \equiv 0 \pmod 5$ has $\ell(v)$ or $\ell(\varphi v)$ in $\{1, 6, 7, 9\}$, a statement about the orbit that Theorem 1.1 does not use.