A lattice parallelepiped with no unimodular triangulation

P. Graham
https://orcid.org/0009-0007-8493-1505

November 7, 2026

Abstract

A lattice polytope has a unimodular triangulation if it can be cut into lattice simplices of the smallest possible volume. Haase, Paffenholz, Piechnik and Santos asked in their 2017 survey whether every lattice parallelepiped, and more generally every lattice zonotope, has one. Such polytopes have the integer decomposition property, and Codenotti and Santos proved that every 3-dimensional lattice parallelepiped can be covered by unimodular simplices. We show that the answer is no in every dimension $d \geq 4$. The 4-dimensional parallelepiped P spanned by e_1, e_2, e_3 and $(1, 2, 3, 5)$ has no unimodular triangulation, and its products with unit cubes give examples in higher dimensions. Its edges at the origin span a cone that Firla and Ziegler showed cannot be triangulated into unimodular cones generated by its Hilbert basis (its minimal generators). Yet the cone of P at each vertex can be so triangulated with lattice points of P: the obstruction is not seen at any vertex. The proof is by hand: points of P with coordinates in $\frac{1}{2}\mathbb{Z}$ and $\frac{1}{4}\mathbb{Z}$ force faces into any unimodular triangulation, until one such point lies in none. Checked certificates show that P has a unimodular cover and is, up to lattice equivalence, the unique 4-dimensional example of determinant at most 5. The question in dimension 3 remains open.

This work has not yet been reviewed by experts. The proofs, the checking programs and the text were produced by AI models (from OpenAI and Anthropic), working under the author's direction.

MSC 2020: 52B20, 52B11, 14M25.
Keywords: unimodular triangulation, lattice parallelepiped, lattice zonotope, unimodular cover, IDP polytope.

1 Introduction

A *lattice polytope* $P \subset \mathbb{R}^d$ is the convex hull of finitely many points of $\mathbb{Z}^d$. A lattice d-simplex is *unimodular* if its vertices $v_0, \ldots, v_d$ satisfy $|\det(v_1 - v_0, \ldots, v_d - v_0)| = 1$, that is, if the edge vectors at one vertex form a basis of $\mathbb{Z}^d$; equivalently, its volume is $1/d!$. A *unimodular triangulation* of P is a triangulation of P into unimodular simplices: a finite set of them whose union is P, any two of which meet in a common face (possibly empty). A *unimodular cover* is a finite set of unimodular simplices whose union is P, with no condition on how they meet. P has the *integer decomposition property* (is *IDP*, also called *normal*) if every lattice point of kP is a sum of k lattice points of P, for every positive integer k. Each property implies the next:

$$\text{unimodular triangulation} \;\Rightarrow\; \text{unimodular cover} \;\Rightarrow\; \text{IDP}.$$

In their survey, Haase, Paffenholz, Piechnik and Santos list these and intermediate properties in a numbered chain that includes the three above, and state that "All the implications (i) $\Rightarrow$ (i+1) are strict" [11, p. 10], citing Bruns and Gubeladze [4]. In a section first added in the third arXiv version (December 11, 2017), they ask [11, §1.5.4]:

> Lattice parallelepipeds and, more generally, lattice zonotopes are known to be IDP. Is it true that they all have unimodular triangulations?

A *zonotope* is a Minkowski sum of segments; a parallelepiped is one with d linearly independent segments in $\mathbb{R}^d$. Ferroni and Higashitani repeat that "it is not known whether zonotopes admit unimodular triangulations in general" [9, §6.1].

Notation. For an integer matrix B with $\det B \neq 0$ we write $P(B) = \{Bt : t \in [0,1]^d\}$ for the lattice parallelepiped spanned by the columns of B, and, in dimension 4, $P[a,b,c,n]$ for $P(B)$ with columns e_1, e_2, e_3 and (a,b,c,n). Its normalized volume is $d!\,|\det B|$; we call $|\det B|$ its *determinant*. Two parallelepipeds are *equivalent* if an affine map $x \mapsto Ux + u$ with $U \in GL_d(\mathbb{Z})$ and $u \in \mathbb{Z}^d$ carries one onto the other; equivalent lattice polytopes have unimodular triangulations or covers together.

1.1 Main results

Theorem 1.1. *The lattice parallelepiped $P = P[1,2,3,5]$, spanned by e_1, e_2, e_3 and $(1,2,3,5)$, has no unimodular triangulation.*

Corollary 1.2. *For every $d \geq 4$ the d-dimensional lattice parallelepiped $P \times [0,1]^{d-4}$ has no unimodular triangulation. Hence, in every dimension $d \geq 4$, neither all lattice parallelepipeds nor all lattice zonotopes have unimodular triangulations.*

Theorems 1.3 and 1.4 and Propositions 1.5 and 1.6 rest on exact computations with checked certificates (Section 6); Theorem 1.1 and Corollary 1.2 do not.

Theorem 1.3. *P has a unimodular cover.*

Theorem 1.4. *Up to equivalence, the 4-dimensional lattice parallelepipeds of determinant at most 8 without a unimodular triangulation are exactly four classes: that of P (which contains $P[1,2,2,5]$), that of $P[1,2,2,7]$ (which contains $P[1,4,3,7]$; see Section 1.2), and those of $P[1,2,3,8]$ and $P[1,3,3,8]$. In particular, P is, up to equivalence, the unique 4-dimensional lattice parallelepiped of determinant at most 5 without a unimodular triangulation.*

A pointed rational cone $C \subset \mathbb{R}^4$ has a *Hilbert basis*, the minimal set of lattice points generating the semigroup $C \cap \mathbb{Z}^4$; a cone is *unimodular* if it is generated by a basis of $\mathbb{Z}^4$. The *vertex cone* of P at a vertex v is the cone generated by $P - v$.

Proposition 1.5. *At each of the 16 vertices v of P, the vertex cone has a triangulation into unimodular cones whose rays pass through lattice points of $P - v$. At $v = 0$ one such triangulation has 56 cones and uses as rays all 19 nonzero lattice points of P, of which 11 are not in the Hilbert basis of the cone.*

Proposition 1.6. *For every odd p with $5 \leq p \leq 31$ the parallelepiped $P[1,2,2,p]$ has no unimodular triangulation.*

1.2 Prior work

Codenotti and Santos proved that "Every 3-dimensional lattice parallelepiped has a unimodular cover" [8, Theorem 1.2] and asked [8, Questions 1.4 and 1.5]: "Do 3-dimensional parallelepipeds have unimodular triangulations?" and "Higher dimensional parallelotopes (affine images of cubes)

are IDP. Do they have unimodular covers?" After stating their cover results (Theorems 1.2 and 1.6 and Corollaries 1.3, 1.7 and 1.8), they add: "We believe that the 3-polytopes in all these statements have unimodular triangulations, but this remains an open question." Corollary 1.2 does not concern dimension 3, and by Theorem 1.3 P does not answer Question 1.5 of [8].

Our example comes from the theory of Hilbert bases of cones. A *Hilbert triangulation* of a cone is a triangulation into unimodular cones generated by elements of its Hilbert basis; Firla and Ziegler call it a Hilbert Partition [10, Proposition 3]. Bruns, Gubeladze and Trung record that Sebő's theorem [12] that every 3-dimensional cone has a unimodular Hilbert triangulation "cannot be extended to dimension ≥ 4, as shown by a counterexample due to Bouvier and Gonzalez-Sprinberg" [7, Section 3.1 and Theorem 3.1.3]. That counterexample is the cone spanned by e_1, e_2, e_3 and $(1, 3, 4, 7)$ [3, Exemple 3.1]; its parallelepiped is equivalent to $P[1, 4, 3, 7]$, the exception of determinant 7 in Theorem 1.4. Writing $C[a_1, \ldots, a_n]$ for the cone spanned by the first $n-1$ unit vectors and $(a_1, \ldots, a_n)$, Firla and Ziegler write: "The saga of counterexamples in this field starts with $C[1, 4, 3, 7]$, the example of Bouvier & Gonzalez-Sprinberg ... from 1992, of a 4-dimensional cone that does not have a Hilbert Partition. A slight simplification yields the following." Their Example 8 follows: "The 4-dimensional cone $C[1, 2, 3, 5]$ does not have a Hilbert Partition" [10]. This is the vertex cone of P at the origin.

Bruns, Gubeladze and Trung also state that "There exists a normal 4-dimensional lattice polytope ... that has no unimodular triangulation" [6, Proposition 1.2.4(c)]. Their polytope is the convex hull of 0 and the Hilbert basis E of the Bouvier–Gonzalez-Sprinberg cone; its only lattice points are 0 and the points of E, and at the vertex 0 a unimodular triangulation of it would give a unimodular Hilbert triangulation of the cone [6, proof of Proposition 1.2.4(c), after Theorem 1.3.1]. That argument does not apply to P: a triangulation of P may use lattice points of P outside the Hilbert basis, and by Proposition 1.5 every vertex cone of P can be triangulated unimodularly with lattice points of P. Theorem 1.1 gives a normal polytope without a unimodular triangulation that is a parallelepiped, hence a zonotope. Appendix C lists the sources we could not read in full.

2 Proof overview

The obstruction in P is not at a vertex (Proposition 1.5), and the step that needs an idea is to find where it sits. Suppose T is a unimodular triangulation of P. Every point $y \in P$ lies in the relative interior of exactly one face G of T, and since G is unimodular, a point with $Ny \in \mathbb{Z}^4$ has barycentric coordinates in G that are positive multiples of $1/N$ (Lemma 3.1). For $N = 2$ this forces a face to be an edge with y as its midpoint, and for most half-integral points of P there is only one pair of lattice points of P with that midpoint: we get twelve edges of T for free. For $N = 4$ there are finitely many candidate faces, listed by hand; two faces of T meet in a common face (Lemma 3.2), and this rules out every candidate that crosses a known face. One quarter-integral point q then forces a tetrahedron on four lattice points near the origin, one of whose edges passes through the center of P, to be a face of T, and a second quarter-integral point r then lies in the relative interior of no possible face. We work in the equivalent parallelepiped $P[1, 2, 2, 5]$, whose lattice points have last coordinate 0 to 5; this coordinate organizes every case split.

3 Two lemmas

In this section Q is a lattice polytope of dimension d and T is a unimodular triangulation of Q. A *face of* T is a face of one of its simplices, vertices included.

Lemma 3.1. *Let $N \geq 1$ and let $y \in Q$ with $Ny \in \mathbb{Z}^d$. Then y lies in the relative interior of exactly one face G of T, and $y = (m_1 g_1 + \cdots + m_k g_k)/N$, where $g_1, \ldots, g_k$ are the vertices of G and $m_1, \ldots, m_k$ are positive integers with sum N. In particular, every lattice point of Q is a vertex of T, and a point y with $2y \in \mathbb{Z}^d \setminus 2\mathbb{Z}^d$ is the midpoint of an edge of T.*

Proof. Since T covers Q, y lies in a simplex $\sigma = \operatorname{conv}(v_0, \ldots, v_d)$ of T; let G be the smallest face of σ containing y, so that $y \in \operatorname{relint} G$. Because σ is unimodular, $v_1 - v_0, \ldots, v_d - v_0$ is a basis of $\mathbb{Z}^d$, so every lattice point has integer barycentric coordinates with respect to σ. If λ are the barycentric coordinates of y, then the lattice point $Ny - (N-1)v_0$ has coordinates $N\lambda - (N-1)(1, 0, \ldots, 0)$, so every $N\lambda_i$ is an integer, and $\lambda_i > 0$ exactly on the vertices of G; put $m_i = N\lambda_i$. If also $y \in \operatorname{relint} G'$ for a face G' of $\sigma' \in T$, then y lies in the common face $F = \sigma \cap \sigma'$, which contains G and G' as the smallest faces of σ and σ' containing y; in the simplex F only one face contains y in its relative interior, so $G = G'$. □

Lemma 3.2. *Let $\operatorname{conv}(S)$ and $\operatorname{conv}(R)$ be faces of T with vertex sets S and R, and suppose that*

$$\sum_{s \in S} p_s\, s = \sum_{r \in R} q_r\, r \quad \text{with integers } p_s, q_r \geq 0 \text{ and } \sum_s p_s = \sum_r q_r > 0.$$

Then $p_s = 0$ for every $s \in S \setminus R$.

Proof. Let $\operatorname{conv}(S) \subseteq \sigma_1$ and $\operatorname{conv}(R) \subseteq \sigma_2$ with $\sigma_1, \sigma_2 \in T$, and let $F = \sigma_1 \cap \sigma_2$, a common face. Then $\operatorname{conv}(S) \cap F$ and $\operatorname{conv}(R) \cap F$ are faces of the simplex F, spanned by the vertices of S and of R that lie in F, and their intersection is the face of F spanned by their common vertices. Hence $\operatorname{conv}(S) \cap \operatorname{conv}(R) = \operatorname{conv}(S \cap R)$. The point $z = \sum_s p_s s / \sum_s p_s$ lies in both, hence in $\operatorname{conv}(S \cap R)$. Since S is affinely independent, the barycentric coordinates $p_s / \sum p_s$ of z in $\operatorname{conv}(S)$ are unique, so they vanish outside $S \cap R$. □

We use Lemma 3.2 to rule out a candidate face $\operatorname{conv}(S)$: we exhibit an identity of this form, with R already known to be a face of T, in which some $s \notin R$ has $p_s > 0$. When R is a single lattice point $x \notin S$, this says that $\operatorname{conv}(S)$ contains x, while a unimodular simplex contains no lattice points other than its vertices.

4 Proof of Theorem 1.1

An equivalent parallelepiped. Let $U \in GL_4(\mathbb{Z})$ have rows $(1,0,0,0)$, $(0,1,0,0)$, $(0,0,-1,1)$, $(0,0,0,1)$. It maps $e_1, e_2, e_3, (1,2,3,5)$ to $e_1, e_2, -e_3, (1,2,2,5)$, so $x \mapsto Ux + e_3$ maps $P[1,2,3,5]$ onto $P' = P[1,2,2,5]$, and it suffices to show that P' has no unimodular triangulation. Writing $x = t_1 e_1 + t_2 e_2 + t_3 e_3 + t_4 (1,2,2,5)$ gives $t_4 = x_4/5$, $t_1 = x_1 - x_4/5$ and $t_i = x_i - 2x_4/5$ for $i = 2, 3$, so $x \in P'$ if and only if

$$0 \leq x_4 \leq 5, \qquad \tfrac{x_4}{5} \leq x_1 \leq 1 + \tfrac{x_4}{5}, \qquad \tfrac{2x_4}{5} \leq x_2, x_3 \leq 1 + \tfrac{2x_4}{5}.$$

For integer $x_4 = k$ this allows $x_1 \in \{0, 1\}$ and $x_2, x_3 \in \{0, 1\}$ if $k = 0$; $x = (1,1,1,k)$ if $k = 1, 2$; $x = (1,2,2,k)$ if $k = 3, 4$; and $x_1 \in \{1, 2\}$, $x_2, x_3 \in \{2, 3\}$ if $k = 5$. So P' has exactly 20 lattice points. With $a, b \in \{0, 1\}$ we name them

$$A_{ab} = (0, a, b, 0), \quad B_{ab} = (1, a, b, 0), \quad C_{ab} = (1, 2+a, 2+b, 5), \quad D_{ab} = (2, 2+a, 2+b, 5),$$

the 16 vertices of P', and the interior points $I_1 = (1,1,1,1)$, $I_2 = (1,1,1,2)$, $I_3 = (1,2,2,3)$, $I_4 = (1,2,2,4)$; the last coordinate of I_k is k. For example $A_{01} = (0,0,1,0)$ and $C_{10} = (1,3,2,5)$. Suppose, for a contradiction, that T is a unimodular triangulation of P'.

Step 1: twelve forced edges. Each row of Table 1 gives lattice points u, w of P' whose sum $2y = u + w$ has an odd coordinate. Then $y \in P'$ is not a lattice point, so by Lemma 3.1 it is the midpoint of an edge of T, whose endpoints are two distinct lattice points of P' with sum $2y$. Their last coordinates add up to the last coordinate of $2y$, which is at most 7, and the table runs through every way to split it; only $u+w$ gives the sum $2y$. Hence each of the twelve segments uw in Table 1 is an edge of T.

Step 2: the tetrahedron $A_{11}I_1I_2I_3$ is a face of T. Let $q = (A_{11} + I_1 + I_2 + I_3)/4$, so that $4q = (3, 5, 5, 6)$; then $q \in P'$ and q is not a lattice point. By Lemma 3.1 the face of T that contains q in its relative interior is $\operatorname{conv}(S)$, where S is a set of lattice points of P' carrying positive integer weights with sum 4 and weighted sum $(3, 5, 5, 6)$. We write a weighted set as a sum: $A_{00}+B_{11}+2I_3$ is $\{A_{00}, B_{11}, I_3\}$ with weights $1, 1, 2$; and $A_{11}\,I_2$ is the face with vertices A_{11} and I_2. There are exactly 21 such weighted sets (Lemma 4.1). For each of the 20 other than $A_{11} + I_1 + I_2 + I_3$, Table 2 gives an identity of Lemma 3.2 with a known face R, either an edge from Step 1 or a lattice point. The coefficients of an identity need not be the weights of S: in the first row $S = A_{00} + B_{11} + 2I_3$, and the identity $2A_{00} + B_{11} + 2I_3 = 2A_{11} + 3I_2$ (both sides have total weight 5) has $p_{A_{00}} = 2 > 0$ with $A_{00} \notin \{A_{11}, I_2\}$, so $\operatorname{conv}(A_{00}, B_{11}, I_3)$ is not a face of T. Hence $\operatorname{conv}(A_{11}, I_1, I_2, I_3)$ is a face of T, and so is its edge I_2I_3, whose midpoint is the center $(1, \frac{3}{2}, \frac{3}{2}, \frac{5}{2})$ of P'.

Step 3: a point that no face can contain. Let $r = (3B_{01} + C_{10})/4$, so that $4r = (4, 3, 5, 5)$; again $r \in P'$ is not a lattice point. The candidate faces containing r in their relative interior are the 17 weighted sets with sum 4 and weighted sum $(4, 3, 5, 5)$ (Lemma 4.1). Table 3 rules out each of them by Lemma 3.2, using an edge from Step 1 or the tetrahedron $F = A_{11}I_1I_2I_3$ of Step 2. Nine of them contain a segment through the center that crosses the edge I_2I_3 of F: for example $B_{01} + C_{10} = I_2 + I_3$ rules out $3B_{01} + C_{10}$ itself. So no face of T contains r in its relative interior. This contradicts Lemma 3.1, and proves Theorem 1.1. □

Lemma 4.1. *The sets of distinct lattice points of P' with positive integer weights summing to 4 and weighted sum $(3, 5, 5, 6)$ are $A_{11} + I_1 + I_2 + I_3$ and the 20 sets in the first column of Table 2. Those with weighted sum $(4, 3, 5, 5)$ are the 17 sets in the first column of Table 3.*

The proof, a short case analysis by the last coordinates, is in Appendix A.

Remark 4.2. The proof uses that two faces of T meet in a common face. A unimodular cover need not satisfy this, and P has one (Theorem 1.3).

Proof of Corollary 1.2. The polytope $Q = P \times [0, 1]^{d-4}$ is the lattice parallelepiped spanned by the vectors $e_1, e_2, e_3, (1, 2, 3, 5)$, padded with zeros, and by $e_5, \ldots, e_d$. Suppose T is a unimodular triangulation of Q. Its faces contained in the 4-dimensional face $P \times \{0\}$ of Q (a facet only when $d = 5$) triangulate that face, so its maximal simplices are 4-dimensional. Each of these is a face of a unimodular simplex of T, so its four edge vectors at a vertex are part of a basis of $\mathbb{Z}^d$; they lie in $\mathbb{Z}^4 \times \{0\}$, so they form a basis of it. So T restricts to a unimodular triangulation of P, contradicting Theorem 1.1. Parallelepipeds are zonotopes, Minkowski sums of segments. □

5 The computer-assisted results

A second proof of Theorem 1.1. A *deletion order* for a lattice polytope Q lists all unimodular simplices in Q so that each one, σ, has a facet F not contained in a facet of Q such that every unimodular simplex in Q with facet F on the other side of the hyperplane spanned by F comes

before σ. A polytope with a deletion order has no unimodular triangulation (Lemma B.1). P has a deletion order of its 3380 unimodular simplices, which proves Theorem 1.1 again.

Theorem 1.3. A subtraction tree (Appendix B) covers P by 2441 of its 3380 unimodular simplices; each of its 33,477 branches ends in an empty polyhedron.

Theorem 1.4. Every lattice parallelepiped is a translate of some $P(B)$. Multiplying B on the left by a matrix in $GL_4(\mathbb{Z})$ brings it to Hermite normal form H: upper triangular, with positive diagonal, $\det H = n$ and $0 \le H_{ij} < H_{jj}$ for $i < j$. This does not change the class of $P(B)$, and neither does permuting the columns of B and changing their signs, which translates $P(B)$. Conversely, an equivalence maps vertices to vertices and edges to edges, and the edge vectors at a vertex of $P(B)$ are the columns of B up to sign; so if $UP(B') + u = P(B)$, then $UB' = BD\Pi$ for a diagonal sign matrix D and a permutation matrix Π, and B' and $BD\Pi$ have the same Hermite normal form. So the classes are the Hermite normal forms up to this relation; the table counts forms and classes. For each $n \le 8$ a list gives one matrix per class and, for each class, either a unimodular triangulation or a deletion order (Appendix B). Determinant 1 gives the unit cube.

determinant n	2	3	4	5	6	7	8
Hermite normal forms	15	40	155	156	600	400	1395
classes	4	4	16	8	26	12	61
without a unimodular triangulation	0	0	0	1	0	1	2

Proposition 1.5. At each vertex v of P, a triangulation of the vertex cone into unimodular cones with rays through lattice points of $P - v$ was found with the SAT solver CaDiCaL [1] and stored; Appendix B describes its two checks. At $v = 0$ the Hilbert basis of the cone consists of $e_1, e_2, e_3, (1, 2, 3, 5)$ and the four interior lattice points of P; the 11 other nonzero lattice points of P, the sums of at least two of the generators, are reducible.

Proposition 1.6. For each odd p from 5 to 31 a deletion order removes all unimodular simplices of $P[1, 2, 2, p]$. The cases $p = 5$ and $p = 7$ are the classes of determinant 5 and 7 in Theorem 1.4, and $P[1, 2, 2, p]$ has a unimodular triangulation for $1 \le p \le 4$ and $p = 6, 8$. This suggests a question.

Question 5.1. Is it true that $P[1, 2, 2, p]$ has no unimodular triangulation for every odd $p \ge 5$?

6 Computational verification

Theorem 1.1 and Corollary 1.2 have hand proofs. A short program re-checks every finite statement in the proof of Theorem 1.1 by brute force, reading Tables 1–3 from this paper's source.

The other results rest on certificates checked in exact integer and rational arithmetic by programs that share no code with the searches that produced the certificates:

- the deletion orders for P, for the four exceptional classes of Theorem 1.4 and for Proposition 1.6, each checked by two separately written programs, one in Python and one in C++, that recompute the lattice points and unimodular simplices of the parallelepiped from its generators;
- the subtraction tree of Theorem 1.3, replayed by one program that recomputes every Farkas certificate and the unimodular simplices of P from its generators;

- the class lists of Theorem 1.4, each checked by two separately written programs that re-enumerate all Hermite normal forms and verify every listed triangulation, one deciding proper intersection by circuits and the other by separating hyperplanes, proposed by linear programming and verified exactly;
- the vertex fans of Proposition 1.5, checked by two separately written programs that recompute the lattice points of P from its generators, one deciding proper intersection by circuits and covering by volumes, the other by Fourier–Motzkin elimination and by neighboring cones; both recompute the Hilbert basis at 0.

The checkers of the hand proof, the deletion orders, the subtraction tree and the vertex fans are also run on deliberately corrupted inputs (a wrong identity, a missing row, a deletion order with its last step moved first, a truncated tree, a fan with one cone dropped or replaced) and reject them. The checks rely on the correctness of the Python and C++ implementations, their compilers and runtimes; there is no formal proof in a proof assistant. All checks run in about half an hour on an ordinary computer, with under 1 GB of memory; the instructions are in the accompanying repository.

Appendix A Tables and the enumeration

The notation is that of Step 2 in Section 4.

Table 1: Step 1: the edge uw, the sum $2y = u + w$, and every split of its last coordinate.

edge	$2y$	splits of the last coordinate
$A_{00}\, I_1$	$(1,1,1,1)$	$0+1$: $I_1 + (0,0,0,0) = I_1 + A_{00}$
$A_{01}\, I_1$	$(1,1,2,1)$	$0+1$: $I_1 + (0,0,1,0) = I_1 + A_{01}$
$A_{10}\, I_1$	$(1,2,1,1)$	$0+1$: $I_1 + (0,1,0,0) = I_1 + A_{10}$
$A_{11}\, I_2$	$(1,2,2,2)$	$0+2$: $I_2 + (0,1,1,0) = I_2 + A_{11}$; $1+1$ needs I_1 twice
$A_{00}\, I_3$	$(1,2,2,3)$	$0+3$: $I_3 + (0,0,0,0) = I_3 + A_{00}$; $1+2$: $I_1 + I_2 = (2,2,2,3) \neq 2y$
$A_{01}\, I_3$	$(1,2,3,3)$	$0+3$: $I_3 + (0,0,1,0) = I_3 + A_{01}$; $1+2$: $(2,2,2,3) \neq 2y$
$A_{10}\, I_3$	$(1,3,2,3)$	$0+3$: $I_3 + (0,1,0,0) = I_3 + A_{10}$; $1+2$: $(2,2,2,3) \neq 2y$
$B_{01}\, I_3$	$(2,2,3,3)$	$0+3$: $I_3 + (1,0,1,0) = I_3 + B_{01}$; $1+2$: $(2,2,2,3) \neq 2y$
$B_{10}\, I_3$	$(2,3,2,3)$	$0+3$: $I_3 + (1,1,0,0) = I_3 + B_{10}$; $1+2$: $(2,2,2,3) \neq 2y$
$I_1\, C_{01}$	$(2,3,4,6)$	$1+5$: $I_1 + (1,2,3,5) = I_1 + C_{01}$; $2+4$: $I_2 + I_4 = (2,3,3,6) \neq 2y$; $3+3$ needs I_3 twice
$I_1\, D_{01}$	$(3,3,4,6)$	$1+5$: $I_1 + (2,2,3,5) = I_1 + D_{01}$; $2+4$: $(2,3,3,6) \neq 2y$; $3+3$ needs I_3 twice
$I_2\, D_{11}$	$(3,4,4,7)$	$2+5$: $I_2 + (2,3,3,5) = I_2 + D_{11}$; $3+4$: $I_3 + I_4 = (2,4,4,7) \neq 2y$

Proof of Lemma 4.1. The lattice points with last coordinate 0 are the A_{ab} and B_{ab}, with last coordinate 5 the C_{ab} and D_{ab}; the first coordinate is 0 on A_{ab}, 1 on B_{ab}, C_{ab} and every I_k, and 2 on D_{ab}.

Sum $(3,5,5,6)$. Let w be the total weight on C and D points. The last coordinates sum to 6, so $w \leq 1$. If $w = 1$, the interior points contribute 1 to the last coordinate, so I_1 has weight 1 and weight 2 lies on A and B points. With a point D_{ab} the weight-2 part has first coordinate 0, so it consists of A points with x_2-sum $2-a$ and x_3-sum $2-b$: this gives $2A_{11}$ with D_{00}; $\{A_{10}, A_{11}\}$ with D_{01}; $\{A_{01}, A_{11}\}$ with D_{10}; and $\{A_{00}, A_{11}\}$ or $\{A_{01}, A_{10}\}$ with D_{11}, five sets. With a point C_{ab} the weight-2 part has first coordinate 1, so it is one A and one B point, with x_2-sum $2-a$ and x_3-sum $2-b$; this gives $1, 2, 2, 4$ sets for $C_{00}, C_{01}, C_{10}, C_{11}$, nine sets. If $w = 0$, the interior weights contribute 6 to the last coordinate. The first coordinates sum to 3, which is the interior weight plus the B weight, so exactly one A point occurs, with weight 1. The interior parts that leave room for

Table 2: Step 2: the weighted sets with sum $(3,5,5,6)$ other than $A_{11}+I_1+I_2+I_3$, a known face R, and an identity of Lemma 3.2.

weighted set S	face R	identity
$A_{00}+B_{11}+2I_3$	$A_{11}\,I_2$	$2A_{00}+B_{11}+2I_3=2A_{11}+3I_2$
$A_{01}+B_{10}+2I_3$	$A_{10}\,I_1$	$A_{01}+2B_{10}+I_3=A_{10}+3I_1$
$A_{10}+B_{01}+2I_3$	$A_{01}\,I_1$	$A_{10}+2B_{01}+I_3=A_{01}+3I_1$
$A_{11}+B_{00}+2I_3$	$A_{00}\,I_1$	$A_{11}+2B_{00}+I_3=A_{00}+3I_1$
$A_{11}+2I_1+I_4$	$A_{00}\,I_3$	$A_{11}+I_1+2I_4=A_{00}+3I_3$
$2A_{11}+I_1+D_{00}$	$I_2\,D_{11}$	$A_{11}+2D_{00}+I_1=D_{11}+3I_2$
$A_{00}+A_{11}+I_1+D_{11}$	$A_{11}\,I_2$	$2A_{00}+D_{11}+I_1=A_{11}+3I_2$
$A_{00}+B_{11}+I_1+C_{11}$	I_3	$C_{11}+I_1=2I_3$
$A_{01}+A_{10}+I_1+D_{11}$	$B_{01}\,I_3$	$A_{01}+D_{11}+I_1=B_{01}+2I_3$
$A_{01}+A_{11}+I_1+D_{10}$	$B_{10}\,I_3$	$A_{11}+D_{10}+I_1=B_{10}+2I_3$
$A_{01}+B_{10}+I_1+C_{11}$	I_3	$C_{11}+I_1=2I_3$
$A_{01}+B_{11}+I_1+C_{10}$	$A_{00}\,I_3$	$A_{01}+C_{10}+I_1=A_{00}+2I_3$
$A_{10}+A_{11}+I_1+D_{01}$	$B_{01}\,I_3$	$A_{11}+D_{01}+I_1=B_{01}+2I_3$
$A_{10}+B_{01}+I_1+C_{11}$	I_3	$C_{11}+I_1=2I_3$
$A_{10}+B_{11}+I_1+C_{01}$	$A_{00}\,I_3$	$A_{10}+C_{01}+I_1=A_{00}+2I_3$
$A_{11}+B_{00}+I_1+C_{11}$	I_3	$C_{11}+I_1=2I_3$
$A_{11}+B_{01}+I_1+C_{10}$	$A_{10}\,I_3$	$A_{11}+C_{10}+I_1=A_{10}+2I_3$
$A_{11}+B_{10}+I_1+C_{01}$	$A_{01}\,I_3$	$A_{11}+C_{01}+I_1=A_{01}+2I_3$
$A_{11}+B_{11}+I_1+C_{00}$	$A_{00}\,I_3$	$A_{11}+C_{00}+I_1=A_{00}+2I_3$
$A_{11}+B_{11}+I_2+I_4$	I_1	$B_{11}+I_2=2I_1$

it are I_4+I_2, I_4+2I_1, $2I_3$, $I_3+I_2+I_1$ and $3I_2$. Matching the second and third coordinates (both 5) gives $A_{11}+B_{11}+I_2+I_4$; $A_{11}+2I_1+I_4$; the four sets $A_{ab}+B_{a'b'}+2I_3$ with $a+a'=b+b'=1$; $A_{11}+I_1+I_2+I_3$; and nothing for $3I_2$, which would need $a=2$. That is seven sets, and 21 in all.

Sum $(4,3,5,5)$. The last coordinates sum to 5, so the total weight on C and D points is at most 1. With one point C_{ab}, the weight-3 rest has first coordinate 3, so it consists of B points, with x_2-sum $1-a$ and x_3-sum $3-b$: C_{00} gives $2B_{01}+B_{11}$; C_{10} gives $3B_{01}$; C_{01} gives $2B_{01}+B_{10}$ and $B_{00}+B_{01}+B_{11}$; C_{11} gives $B_{00}+2B_{01}$; five sets. With one point D_{ab}, the rest has first coordinate 2, so it is one A point and B weight 2, with x_2-sum $1-a$ and x_3-sum $3-b$: D_{00} gives two sets, D_{10} one, D_{01} five and D_{11} two, ten sets in all, as listed in Table 3. With no C or D point, the interior weights contribute 5 to the last coordinate and the first coordinates force the interior weight plus the B weight to be 4, so no A point occurs. The interior parts are I_4+I_1, I_3+I_2, I_3+2I_1, $2I_2+I_1$ and I_2+3I_1; the second coordinate 3 excludes I_3+2I_1 and I_2+3I_1, $2I_2+I_1$ would need a B point with third coordinate 2, and I_4+I_1 and I_3+I_2 each need $2B_{01}$. That is two sets, and 17 in all. □

Table 3: Step 3: the weighted sets with sum $(4,3,5,5)$, a known face R, and an identity of Lemma 3.2.

weighted set S	face R	identity
$3B_{01} + C_{10}$	$A_{11}\, I_1\, I_2\, I_3$	$B_{01} + C_{10} = I_2 + I_3$
$A_{00} + 2B_{01} + D_{11}$	$A_{11}\, I_1\, I_2\, I_3$	$A_{00} + D_{11} = I_2 + I_3$
$A_{01} + 2B_{01} + D_{10}$	$A_{11}\, I_1\, I_2\, I_3$	$A_{01} + D_{10} = I_2 + I_3$
$A_{10} + 2B_{01} + D_{01}$	$A_{11}\, I_1\, I_2\, I_3$	$A_{10} + D_{01} = I_2 + I_3$
$A_{11} + 2B_{01} + D_{00}$	$A_{11}\, I_1\, I_2\, I_3$	$A_{11} + D_{00} = I_2 + I_3$
$B_{00} + 2B_{01} + C_{11}$	$A_{11}\, I_1\, I_2\, I_3$	$B_{00} + C_{11} = I_2 + I_3$
$2B_{01} + B_{10} + C_{01}$	$A_{11}\, I_1\, I_2\, I_3$	$B_{10} + C_{01} = I_2 + I_3$
$2B_{01} + B_{11} + C_{00}$	$A_{11}\, I_1\, I_2\, I_3$	$B_{11} + C_{00} = I_2 + I_3$
$2B_{01} + I_1 + I_4$	$A_{11}\, I_1\, I_2\, I_3$	$I_1 + I_4 = I_2 + I_3$
$2B_{01} + I_2 + I_3$	$I_1\, C_{01}$	$B_{01} + I_2 + 2I_3 = C_{01} + 3I_1$
$A_{00} + B_{01} + B_{11} + D_{01}$	$A_{01}\, I_1$	$2A_{00} + 3B_{11} + D_{01} = A_{01} + 5I_1$
$A_{01} + B_{00} + B_{01} + D_{11}$	$I_1\, D_{01}$	$A_{01} + 2B_{00} + B_{01} + 3D_{11} = 2D_{01} + 5I_1$
$A_{01} + B_{00} + B_{11} + D_{01}$	$B_{01}\, I_3$	$3A_{01} + B_{00} + 4B_{11} + 3D_{01} = 6B_{01} + 5I_3$
$A_{01} + B_{01} + B_{10} + D_{01}$	$A_{11}\, I_1\, I_2\, I_3$	$A_{01} + 2B_{10} + D_{01} = 3I_1 + I_2$
$A_{01} + B_{01} + B_{11} + D_{00}$	$I_1\, D_{01}$	$A_{01} + 3B_{11} + 2D_{00} = D_{01} + 5I_1$
$A_{11} + B_{00} + B_{01} + D_{01}$	$A_{01}\, I_1$	$3A_{11} + 2B_{00} + B_{01} + D_{01} = 2A_{01} + 5I_1$
$B_{00} + B_{01} + B_{11} + C_{01}$	$B_{01}\, I_3$	$B_{00} + 4B_{11} + 3C_{01} = 3B_{01} + 5I_3$

Appendix B The certificates

Deletion orders.

Lemma B.1. *Let T be a unimodular triangulation of a lattice polytope Q, let $\sigma \in T$, and let F be a facet of σ not contained in a facet of Q. Then some $\tau \in T$ has F as a facet and lies on the other side of the hyperplane spanned by F. Hence a polytope with a deletion order has no unimodular triangulation.*

Proof. A point $x \in \operatorname{relint} F$ lies in the interior of Q, since otherwise a supporting hyperplane of Q through x would contain F, and F would lie in a facet of Q. Points near x on the other side of F lie in Q and are covered by the finitely many simplices of T, so one of them, τ, contains such points arbitrarily close to x, and x itself. Then $\sigma \cap \tau$ is a common face containing x, hence F; it is not σ, since τ has points on the other side, so F is a facet of τ, and τ lies on the other side. If Q had a deletion order and a unimodular triangulation $T \neq \emptyset$, the first simplex of T in the order would have such a τ before it, a contradiction. □

The subtraction tree. We cover the interior of P recursively. Given an open polyhedron R and a unimodular simplex S in P with facet inequalities $f_0, \dots, f_4 \geq 0$, the part of R outside S is, up to finitely many hyperplanes, the union of the open polyhedra $R \cap \{f_0 > 0, \dots, f_{i-1} > 0,\ f_i < 0\}$ for $i = 0, \dots, 4$; we recurse on each. A branch stops when its polyhedron $\{x : a_k \cdot x < b_k \text{ for all } k\}$ is empty, which an exact Farkas certificate proves: numbers $y_k \geq 0$, not all zero, with $\sum_k y_k a_k = 0$ and $\sum_k y_k b_k \leq 0$. The tree for P starts from its interior, so the interior of P minus the union of the subtracted simplices lies in finitely many hyperplanes; the union, which is closed, is P.

Triangulations of the classes. A list of $24n = 4! \cdot n$ distinct unimodular simplices, the normalized volume of $P(B)$, with vertices among the lattice points of $P(B)$, is a triangulation when any two meet in a common face: their interiors are then disjoint, their volumes add up to that of $P(B)$, and the closed union is $P(B)$. One checker decides this by circuits: two simplices fail to meet in a common face exactly when some minimal affinely dependent set of their vertices has the points with positive coefficients in one simplex and those with negative coefficients in the other. The other checker verifies, in exact arithmetic, a hyperplane through their common vertices that separates them, found by linear programming.

Vertex fans. Let v be a vertex of P, let $g_1, \dots, g_4$ be the edge vectors of P at v (the columns of B up to sign), so that the vertex cone is $C_v = \operatorname{cone}(g_1, \dots, g_4)$, and let $h(x)$ be the sum of the coordinates of x in the basis $g_1, \dots, g_4$. Each stored cone is spanned by $p_1 - v, \dots, p_4 - v$ for lattice points p_i of P with $|\det(p_1 - v, \dots, p_4 - v)| = 1$, and lies in C_v. Both checkers verify this from the lattice points of P, recomputed from its generators. The first checks that the cones meet pairwise in common faces, by the circuit test on the simplices $\operatorname{conv}(0, p_1 - v, \dots, p_4 - v)$, and that

$$\sum \frac{1}{h(p_1 - v)\, h(p_2 - v)\, h(p_3 - v)\, h(p_4 - v)} = |\det B|,$$

summed over the cones. Up to one common factor, the left side is the total volume of the sections of the cones by the hyperplane $h = 1$, and the right side is the volume of the section of C_v. With disjoint interiors, the closed sections then fill the section of C_v, so the cones fill C_v. The second checker shows by Fourier–Motzkin elimination that no point of two cones has a positive coordinate, in the basis of one of them, at a ray the other lacks; so the cones meet in common faces. It then

checks that every facet of a cone that is not contained in a facet of C_v is a facet of another cone on its other side. Then the cones fill C_v: otherwise a generic segment from the interior of a cone to an interior point of C_v outside their union would leave the union through such a facet. Both checkers compute the Hilbert basis of C_0 from the lattice points of P: its elements lie in P, and a lattice point of P is reducible exactly when it is the sum of two nonzero lattice points of P.

Appendix C Sources not read in full

The thesis of von Thaden [13], the paper [4] and the published edition of the book [5] of Bruns and Gubeladze (we read a 2006 draft of the book) are all cited in [11], whose authors posed the question after them. Sebő's paper [12] is cited only for its theorem on 3-dimensional cones, as reported in [7]. For Blanco's chapter [2] we read its title and its zbMATH summary (Zbl 1428.52019), which is about covers: "In this paper I will study a 3-parameter family of lattice parallelepipeds. I will study the behavior of the lattice with respect to the parameters and prove that three 2-parameter subfamilies have unimodular covers."

Appendix D Correction

November 7, 2026. The first version's proof of Corollary 1.2 called $P \times \{0\}$ a facet of $P \times [0,1]^{d-4}$. It is a 4-dimensional face, and a facet only when $d = 5$. The proof now says so, and its basis argument now refers to the maximal 4-dimensional simplices of the restricted triangulation: a unimodular triangulation of the product restricts to one of P. The argument, the theorems and the checks are unchanged.

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