Symmetric chains in noncrossing partition lattices

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Abstract

Every noncrossing partition lattice of a finite well-generated complex reflection group admits a symmetric chain decomposition. We give a computer-assisted proof for the exceptional types H_4, G_{33}, G_{34}, E_6, E_7, and E_8, completing the saturated-chain assertion asked by Henri Mühle. Simion and Ullman, Reiner, Bessis and Corran, and Mühle established the other irreducible cases. Explicit chain certificates are checked against exact reflection-group models; exhaustive rank descent certifies that their graphs are the full absolute-order intervals. Reflection-preserving automorphisms and a constructive product decomposition give every Coxeter element and every reducible group. Lean verifies the general interval-closure, product, and transfer lemmas. The finite group calculations and the published classification and decomposition results remain external inputs.

1 Introduction

A noncrossing partition of a circularly ordered set groups its points into blocks whose convex hulls do not cross. Ordering these partitions by refinement gives the classical noncrossing partition lattice. Its generalization to reflection groups replaces refinement by absolute order: an element lies below another when a shortest reflection factorization can be extended to one for the latter. We prove the following statement; the terminology is made precise below. The partition it describes is a symmetric chain decomposition.

Theorem 1.1. *Let W be a finite well-generated complex reflection group of essential rank n. For each irreducible factor choose any Coxeter element, and let c be their product. The noncrossing partition lattice* $\mathrm{NC}(W, c)$ *has a partition into nonempty saturated chains whose bottom and top ranks sum to n. The trivial group, of rank zero, is included.*

The new contribution is the chain decomposition for the six exceptional types named in the abstract. The main difficulty is the compatibility of chains across every rank: rank symmetry, unimodality, and the strong Sperner property do not supply such chains. Mühle's work already established those properties for all well-generated groups [11, Theorem 1.4]. Here explicit partitions supply the missing chain decompositions. Their discovery uses Griggs's matching construction, iterated from the middle ranks as in Tomon's work [8, 18]. Their verification also establishes the

complete interval and its covers, rather than relying on a graph with the expected number of vertices.

The question and earlier work

Henri Mühle asks [11, Question 1.1]:

> Does the lattice of W-noncrossing partitions always admit a symmetric decomposition into saturated chains or Boolean lattices for every well-generated complex reflection group W?

Theorem 1.1 answers its chain part. The Boolean part is not settled here.

Rodica Simion and Daniel Ullman constructed symmetric decompositions for classical noncrossing partitions [17]. Victor Reiner proved the type-B chain assertion [13, Theorem 13]; Patricia Hersh's local symmetric-group actions give the stronger Boolean decomposition in type B [10, Theorems 5 and 7]. David Bessis and Ruth Corran identify the $G(d, 1, m)$ intervals with type B for $d \geq 2$ [3]. Mühle proves the Boolean assertion for $G(d, d, m)$ and the chain assertion for the small exceptional types, leaving the six treated here [11, Theorem 1.3 and Proposition 4.2]. He also records an earlier unpublished type-D chain construction by Eleni Tzanaki [11, introduction, after Theorem 1.3].

The reduction uses the classification of Geoffrey Shephard and John Todd [16]. The exact models use the Coxeter representations described by Drew Armstrong [1], the invariant degrees tabulated by David Bessis [2], and the Coxeter–Todd root coordinates and named generators described by Manabu Oura and Jiro Sekiguchi [12], building on John Conway and Neil Sloane [5]. The passage between all choices of Coxeter element uses the theorem of Victor Reiner, Vivien Ripoll, and Christian Stump [14].

Recent related work includes Michael Dougherty and Gina Root's symmetric chain results under conditions on planar hull configurations [6, Theorem A], and Alice Gao, Xin-Bei Liu, Arthur Yang, and James Zhao's interlacing symmetric decompositions of order-complex h-polynomials [7, Theorem 1.5]. The former concerns a geometric class of partition lattices; the latter decomposes polynomials rather than the elements of an absolute-order interval. These results address different extensions of classical noncrossing-partition structure. For general graded posets, Nate Harman, Rylan Rosas, Charles Troutman III, and Chris Wu introduce strong $\mathfrak{sl}_2$-actions and conjecture their equivalence with symmetric chain decompositions [9, Definition 1.1 and Conjecture 1.2]. The certificates here give the chains directly and do not require that conjecture.

The proof combines exact finite computations with the general lemmas checked in Lean; Section 6 describes their relation.

The proofs, the Lean formalization, the certificate checkers and the text were produced by AI models (from OpenAI and Anthropic), working under the author's direction. This computer-assisted work has not yet been reviewed by experts.

2 Definitions and proof overview

We recall the conventions and then describe the argument.

A *complex reflection* on a finite-dimensional complex vector space V is a nonidentity finite-order linear transformation fixing a hyperplane pointwise. A finite complex reflection group is a finite subgroup of $\mathrm{GL}(V)$ generated by reflections. Its *essential rank* is $\dim V - \dim \mathrm{Fix}(W)$. It is *well-generated* if it can be generated by that many reflections. A fixed summand can be removed without changing any reflection length below.

Let T be the set of all reflections in W. It is closed under inversion and conjugation. The *reflection length* $\ell_T(w)$ is the smallest number of factors from T in a product equal to w. The *absolute order* and its interval below c are

$$u \le_T v \iff \ell_T(u) + \ell_T(u^{-1}v) = \ell_T(v),$$
$$\mathrm{NC}(W, c) = [1, c]_T.$$

The rank of an element in this interval is $\ell_T(w)$.

For the essential rank-n representation, the algebra of invariant polynomials $\mathbb{C}[V]^W$ is a polynomial algebra on n homogeneous generators. Their degrees $d_1 \le \cdots \le d_n$ are independent of the choice of homogeneous generators and are called the *invariant degrees* of W. For irreducible W, let $h = d_n$ be the largest invariant degree. A vector is *regular* if it lies outside every reflecting hyperplane. A *Coxeter element* here means a regular element of order h: it has an eigenvector that is regular, with eigenvalue of order h. This includes every regular element of order h, not just one conjugacy class. For a reducible group the choice is componentwise, as in Theorem 1.1.

For example, $\mathrm{NC}(S_3, (123))$ has rank sizes $1, 3, 1$. A maximal chain $1 < (12) < (123)$, together with the two singleton chains (13) and (23), is a symmetric chain decomposition. The maximal chain has endpoint ranks $0, 2$, and each singleton has rank 1.

Definition 2.1. In a finite graded poset of rank n, a *saturated chain* has a cover relation between each consecutive pair. It is *symmetric* if its endpoint ranks sum to n. A *symmetric chain decomposition* is a partition of the elements into nonempty symmetric saturated chains.

The proof has three parts. First, published decompositions leave only six irreducible types. For one Coxeter element in each type, the supplied certificates list the chains by exact matrix labels. Second, a complete reflection census and exhaustive downward reflection steps identify the certified graph with $[1, c]_T$. The key observation is that, on a reflection walk of length n from c to 1 with $\mathrm{rank}(I - c) = n$, the quantity $\mathrm{rank}(I - w)$ must drop at every step. This turns interval completeness into a finite closure check. Finally, reflection-preserving automorphisms transport the chains to every Coxeter element, and a grid construction combines factor chains for reducible groups.

The argument does not assume that reflection length equals fixed-space codimension on the entire complex reflection group. It proves that equality only for the certified interval.

3 Absolute-order intervals

The finite criterion below applies in any faithful finite-dimensional representation.

Put $q(w) = \mathrm{rank}(I - w)$. The identity

$$I - uv = (I - u) + u(I - v)$$

implies $q(uv) \le q(u) + q(v)$. In particular

$$q(w) \le \ell_T(w), \qquad q(w) \le q(wt) + 1 \quad (t \in T). \tag{1}$$

The second inequality follows by applying subadditivity to $w = (wt)t^{-1}$. A *reflection walk* has steps $w \longmapsto wt$ with $t \in T$.

Lemma 3.1. *Suppose $q(c) = n$ and c has a reflection word of length n. Then $\ell_T(c) = n$. An element x belongs to $[1, c]_T$ if and only if it lies on a reflection walk from c to 1 of length n. Every step of every such walk lowers q by one. At any vertex x on the walk, $q(x) = \ell_T(x)$.*

Proof. The word and (1) give $n = q(c) \le \ell_T(c) \le n$. If $x \le_T c$, reduced words for $x^{-1}c$ and x, removed in reverse order, give a right-reflection walk $c \to x \to 1$ of total length n. Conversely, if such a walk takes k steps to reach x and $n-k$ more to reach 1, then

$$\ell_T(x^{-1}c) \le k, \qquad \ell_T(x) \le n-k.$$

The triangle inequality for reflection length forces their sum to be n, so $x \le_T c$.

The n steps lower q from n to zero. By (1) each step lowers it by at most one. The sum of these decreases is n, so each is one. Thus $q(x) = n-k$. Since $q(x) \le \ell_T(x) \le n-k$, equality holds. □

Proposition 3.2 (Finite interval criterion)**.** *Suppose T is the entire reflection set, $q(c) = n$, and a reflection word of length n for c is supplied. Let S be a finite set of distinct elements of W containing c and 1. Suppose that:*

1. *each $x \in S$ lies on a reflection walk $c \to x \to 1$ of total length n;*
2. *for each $x \in S$ and $t \in T$, if $q(xt) = q(x) - 1$, then $xt \in S$.*

Then $S = [1, c]_T$, and q is its absolute-order rank. Its downward covers are exactly the pairs $x \longmapsto xt$ with $x, xt \in S$ and $q(xt) = q(x) - 1$.

Proof. The first condition and Lemma 3.1 give $S \subseteq [1, c]_T$ and $q = \ell_T$ on S. If $y \in [1, c]_T$, choose a length-n walk through y. Every step lowers q by one. Starting at $c \in S$, the second condition keeps that walk in S; hence $y \in S$.

For a pair in the asserted downward edge set, write $y = xt$. The equality $\ell_T(y) = \ell_T(x) - 1$ and the fact that $y^{-1}x = t^{-1}$ is a reflection give $y <_T x$. Their ranks differ by one, so this is a cover. Conversely, if $y <_T x$, a reduced word for $y^{-1}x$ extends a reduced word for y to one for x. Its successive prefixes give all intervening absolute ranks. They also lie below c, by extending further to a reduced word for c. If $y <_T x$ is a cover, there can therefore be only one added reflection. This is precisely a downward reflection step of rank one. □

The criterion certifies vertices and covers together. Checking only cardinality, or checking chains on two copies of the same incomplete graph, would not establish either conclusion.

4 The exceptional cases

The six certificates use the exact reflection-group models specified in Appendix A. We explain why their checks establish Proposition 3.2. The complete chain partitions and reflection words are in the archived verification package.

The chains use Griggs's matching construction [8] in the iterative form described by Tomon [18, §§3–4]. It starts with singleton chains in the middle rank when the total rank is even, or a perfect matching between the two middle ranks when it is odd. At each stage, a bipartite perfect matching either extends a chain at both ends by one cover or leaves it finished. Two copies of each active chain, joined by an identity edge, allow these choices in a single matching problem.

4.1 Reflections and Coxeter elements

The root formulas provide a candidate reflection set. Its completeness is checked independently of the formula: the checker computes all conjugacy classes of the generated group, tests $\operatorname{rank}(I - g)$ on every class representative, and requires every member of every rank-one class to occur in the

candidate set. Rank is constant on a conjugacy class. The class sizes must sum to the group order. Thus this step establishes the entire reflection set, subject to the exact group algorithms described in Section 6. In these six groups the reflections are involutions; no such restriction is imposed on the published family inputs.

The selected element c is supplied as a word in n reflections. The checker verifies $q(c) = n$, its order h, and an exact eigenvector for $e^{2\pi i/h}$ that is fixed by none of the reflections. This last test puts the eigenvector outside every reflecting hyperplane, so establishes regularity. The invariant degrees, and hence the relevant values of h, are taken from [2, Table 2]; they are not inferred from an observed group order. Table 1 records the parameters and sizes of the certificates.

4.2 Interval and chain verification

For each group the interval generator starts from c and tests every right multiplication by a reflection, retaining exactly the rank descents. It does not read the chain partition. A separately written semantic checker reconstructs the supplied vertices and computes ranks by its own exact row elimination. For every vertex x and every reflection t, it checks the downward-closure condition and requires that the recorded edges are exactly all resulting pairs.

The checker also verifies paths from c to every vertex and from every vertex to 1. Every edge lowers q by one, so these paths have total length n. Strictly decreasing ranks make the path verification noncircular. All hypotheses of Proposition 3.2 are therefore checked. The resulting graph is the full absolute-order interval with its actual covers.

The chain files use the same exact basis-image labels. Two independent chain checkers, written in Python and JavaScript, require each chain to be nonempty, every vertex to occur exactly once, consecutive vertices to be recorded covers of successive ranks, and endpoint ranks to sum to n. These conditions are the definition of a symmetric chain decomposition.

Proposition 4.1. *For each of $H_4, G_{33}, G_{34}, E_6, E_7, E_8$, the supplied certificates give a symmetric chain decomposition of* $\mathrm{NC}(W, c)$ *for the chosen Coxeter element c.*

Proof. The exact checks just described establish the hypotheses of Proposition 3.2 and every condition in the definition of a symmetric chain decomposition. All six certificates pass these checks. □

5 From six types to all groups

We assemble the finite inputs with the published cases, and prove the two elementary transfer steps needed for the full statement.

5.1 Irreducible groups

Write $G(r, p, m)$ for the monomial group whose nonzero entries are rth roots of unity and whose product of nonzero entries is an (r/p)th root of unity, where $p \mid r$. The well-generated filter of the Shephard–Todd classification [16], as stated in [11, §2.2], leaves the families and exceptional types below. The classification and filter are published inputs, not consequences of the certificate computations.

The family $G(1, 1, m)$ is S_m, with essential rank $m - 1$ in its permutation representation. Its noncrossing interval is the classical lattice, and the result is Simion–Ullman's theorem [17, Theorem 2], restated in [11, Theorem 3.6 and Corollary 3.7]. For $d \geq 2$, Bessis–Corran's remark at the end of §7 and their paragraph after Corollary 8.2 identify $\mathrm{NC}(G(d, 1, m))$ with the type-B interval [3]. Reiner's Theorem 13 supplies its chain decomposition [13]. For $d, m \geq 2$, the $G(d, d, m)$ case

is [11, Theorem 1.3]. The reducible parameter $G(2,2,2)$ is also covered by the product argument below. A symmetric Boolean piece is refined into symmetric chains by the product construction of Lemma 5.1; this explains the Boolean-to-chain implication used in the cited family cases [11, Proposition 2.1].

There are 26 exceptional well-generated types, with indices

$$4, 5, 6, 8, 9, 10, 14, 16, 17, 18, 20, 21, 23, 24, 25, 26, 27, 28, 29, 30, 32, 33, 34, 35, 36, 37.$$

Of these, Proposition 4.2 of [11] covers the twenty types

$$4, 5, 6, 8, 9, 10, 14, 16, 17, 18, 20, 21, 23, 24, 25, 26, 27, 28, 29, 32.$$

The remaining six are exactly those in Proposition 4.1. In particular G_{32} has rank four and belongs to the cited cases. This proves the chain assertion for one Coxeter element in every irreducible type. Rank-one groups are cyclic; all their nonidentity elements are reflections, and their interval is the single chain $1 < c$. The rank-zero interval is a singleton.

5.2 Every choice of Coxeter element

Theorem 1.1 allows all Coxeter elements in the regular-element sense. By Reiner–Ripoll–Stump [14, Proposition 1.4 and Corollary 1.5], for any two Coxeter elements c, c' of an irreducible well-generated group there is a group automorphism ϕ with $\phi(T) = T$ and $\phi(c) = c'$. Applying ϕ to a shortest reflection word, and then doing the same with ϕ^{-1}, proves

$$\ell_T(\phi(w)) = \ell_T(w).$$

The defining equality for absolute order now gives

$$u \le_T v \iff \phi(u) \le_T \phi(v).$$

Thus ϕ restricts to a rank-preserving order isomorphism $[1, c]_T \to [1, c']_T$. An order isomorphism preserves covers, partitions, and endpoint ranks; it carries a symmetric chain decomposition to one for c'. This uses a reflection-preserving automorphism, not an assertion that all Coxeter elements are conjugate inside W.

5.3 Products of groups and chains

For the essential product representation $W = W_1 \times W_2$, fixed-space codimensions add. A reflection of W is consequently exactly $(t, 1)$ or $(1, u)$ with t or u a reflection in its factor. Projecting any reflection word and concatenating shortest factor words give the two inequalities proving

$$\ell_T((x, y)) = \ell_{T_1}(x) + \ell_{T_2}(y). \tag{2}$$

For each factor, the triangle inequality says

$$\ell_{T_i}(x_i) + \ell_{T_i}(x_i^{-1} c_i) \ge \ell_{T_i}(c_i).$$

By (2), the absolute-order equality for the product holds exactly when both factor inequalities are equalities. The same calculation for two arbitrary product elements shows that their absolute order is coordinatewise. Hence

$$[1, (c_1, c_2)]_T = [1, c_1]_{T_1} \times [1, c_2]_{T_2} \tag{3}$$

as ranked posets, with ranks added.

The standard product construction is due to de Bruijn, van Ebbenhorst Tengbergen, and Kruyswijk [4]; see also Srinivasan's account [15, §1].

Lemma 5.1. *The product of two finite graded posets with symmetric chain decompositions has a symmetric chain decomposition.*

Proof. Partition the product into products of factor chains. Take chains with bottom ranks p, q and lengths in steps a, b. Their product is a grid $[0, a] \times [0, b]$ whose element (i, j) has rank $p+q+i+j$. For each integer $0 \le k \le \min(a, b)$ take

$$(k, 0), (k, 1), \ldots, (k, b-k), (k+1, b-k), \ldots, (a, b-k). \tag{4}$$

Empty final portions are omitted. A vertex (i, j) lies on exactly the chain with label $\min(i, b-j)$: when $i \le b-j$ it is on the vertical portion, and when $i > b-j$ it is on the horizontal portion. These chains therefore partition the grid.

Each step increments exactly one coordinate and is a cover in its factor, so is a cover in the product poset. The endpoint ranks are $p+q+k$ and $p+q+a+b-k$. Their sum is $(2p+a)+(2q+b)$, the sum of the factor ranks by symmetry of the factor chains. Thus every chain is symmetric in the product. □

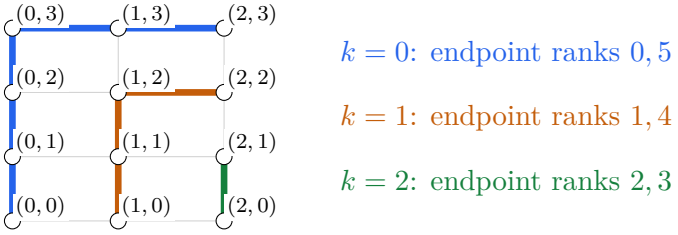


Figure 1: The construction (4) for a product of chains of lengths two and three. The coordinate rank is $i + j$; every colored chain has endpoint-rank sum five. The placement shows the grid coordinates rather than putting equal ranks on horizontal lines.

Proof of Theorem 1.1. The classification reduction, Proposition 4.1, and the automorphism argument give the assertion for every irreducible factor and every Coxeter choice. Equations (2) and (3) identify a reducible group's interval with the ranked product of its factor intervals. Repeated application of Lemma 5.1 gives the required decomposition. The empty product is the rank-zero singleton, whose singleton chain is symmetric. □

6 Computational verification

The formal interval lemma expresses the closure argument: rank-decreasing edges and word witnesses identify the whole interval.

The Lean theorem `IntervalCertificate.exact_interval` proves the interval equality in the following exact form. Let E be a relation on a type V, let $q : V \to \mathbb{N}$, let S be a predicate on V, let $c, z \in V$ and $n \in \mathbb{N}$. Write $x \xrightarrow[E]{k} y$ for an E-walk of length k, and put

$$J_n(x) \iff \exists k, l \in \mathbb{N}, \quad k + l = n, \quad c \xrightarrow[E]{k} x \xrightarrow[E]{l} z.$$

Its statement is

$$\begin{gathered}(\forall x, y,\ E(x, y) \Rightarrow q(x) \le q(y) + 1), \qquad q(c) = n, \quad q(z) = 0, \quad S(c), \\ (\forall x, y,\ S(x) \wedge E(x, y) \wedge q(y) + 1 = q(x) \Rightarrow S(y)), \\ (\forall x,\ S(x) \Rightarrow J_n(x)) \quad \Longrightarrow \quad \forall x,\ S(x) \iff J_n(x).\end{gathered}$$

Here E is right multiplication by a reflection, $z = 1$, and $q(w) = \operatorname{rank}(I - w)$. Lemma 3.1 identifies J_n with $[1, c]_T$; the two certificate conditions in Proposition 3.2 supply the word witnesses and closure. The cover argument remains the written proof of that proposition.

Lean also checks the product and transport constructions, the exceptional-type table partition, and finite products including the empty product. The statement of Theorem 1.1 is also recorded as a definition, using reflection length and actual order-theoretic covers. Its two predicates require finite well-generated complex reflection data and componentwise regular elements of largest invariant degree, as defined in Section 2. The theorem combines the Lean lemmas, finite computations and published inputs through the written reduction above. Reproduction commands and the arithmetic, group algorithms, certificate tests and formal dependencies are described in the companion verification package.

Appendix A Exact reflection-group models

The coordinates below specify the representations used for the finite checks.

For H_4, E_6, E_7, and E_8 we use their geometric Coxeter representations [1, §2.1 and Figure 2.1]. Let s_i be the simple reflection corresponding to a simple root α_i. For $i \neq j$, the Coxeter parameter m_{ij} is the order of $s_i s_j$; set $m_{ii} = 1$. The form has $B(\alpha_i, \alpha_j) = -\cos(\pi/m_{ij})$, in particular $B(\alpha_i, \alpha_i) = 1$, and the simple reflection is $v \mapsto v - 2B(\alpha_i, v)\alpha_i$. The H_4 diagram is the path with edge labels $5, 3, 3$. The E_n diagram has branching-arm lengths $2, 1, n-4$. Closing the simple roots under these reflections gives an exact finite root set; the action on it is faithful because it spans the representation. A product of the simple reflections is the chosen Coxeter element.

For the two nonreal types, put $\omega = e^{2\pi i/3}$ and $\theta = 1 + 2\omega$. The root vectors are the coordinate permutations of

$$\pm\theta(\omega^a, -\omega^b, 0, 0, 0, 0) \quad \text{and} \quad \pm(\omega^{a_1}, \ldots, \omega^{a_6}), \qquad \sum_i a_i \equiv 0 \pmod 3,$$

where all exponents lie in $\{0, 1, 2\}$. These are the Coxeter–Todd coordinates of [12, §2], originating in Conway–Sloane [5]. Each vector v has squared norm 6. In row-vector convention its reflection matrix is

$$s_v = I - \frac{\overline{v}^t v}{3}.$$

The six named reflections $P_1, P_2, P_3, Q_1, R_1, R_2$ generate G_{34}; omitting P_3 gives G_{33} [12, §2 and §6.1]. The fixed hyperplanes, in this order, are

$$x_2 = x_3, \quad x_3 = x_4, \quad x_4 = x_5, \quad x_1 = x_2, \quad x_1 = \omega x_2, \quad \sum_i x_i = 0.$$

These hyperplanes specify the correspondence with the named generators, rather than merely identifying a group by its order. The G_{33} model retains a fixed line in six coordinates; its essential rank is five. This fixed summand changes neither q nor reflection length.

Each model acts by permutations of its exact root list. A vertex label is the list of images of an ordered spanning basis, expressed as root indices. It determines a unique matrix. The semantic checker reconstructs that matrix, checks the action on every root, and verifies membership in the generated permutation group.

Appendix B Exceptional interval sizes

Table 1 lists the sizes of the six verified decompositions.

Table 1: The new finite chain certificates. Counts describe the checked data; they are not assumptions in the interval-completeness argument.

Type	Rank n	h	Vertices	Covers	Chains
$H_4 = G_{30}$	4	30	280	1,050	158
G_{33}	5	18	308	1,365	123
G_{34}	6	42	1,584	8,932	700
$E_6 = G_{35}$	6	12	833	4,284	351
$E_7 = G_{36}$	7	18	4,160	26,208	1,470
$E_8 = G_{37}$	8	30	25,080	188,100	9,518

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