Fathi's consecutive-integers bound does not extend to multitwists

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Abstract

On the sphere with five boundary components, a fixed positive multitwist can have exceptional exponents arbitrarily far apart, even when the base map is pseudo-Anosov. More precisely, we give curves a, c, d such that, for every $k \geq 1$, the multitwist $t = T_a T_c$ and $g_k = T_a^{-1} T_c^{-(k+1)} T_d$ satisfy Ivanov's filling hypothesis (the component orbits together fill the surface), while tg_k and $t^{k+1} g_k$ fix essential curves. Their exceptional exponents are k apart. Thus Fathi's consecutive-integers bound for a single Dehn twist does not extend to the multitwists in Ivanov's question. Long and Morton proved finiteness of the exceptions for a single twist about an essential curve with a pseudo-Anosov base on a closed orientable surface; on closed surfaces, Fathi confined the exceptional exponents to seven consecutive integers whenever the twisting curve's orbit fills the surface, for any base mapping class. We answer Ivanov's consecutive-integers question for this surface; examples with reducible bases are also given. Penner's construction supplies the pseudo-Anosov bases, and a pants decomposition proves filling. The algebra is verified in Lean and the finite curve and topology calculations by independent exact programs. The weaker question about the number of exceptions remains open.

1 Introduction

Let S be a compact orientable surface, possibly with boundary, and let $\mathrm{Mod}_S = \pi_0(\mathrm{Diff}(S))$ be its mapping class group. A mapping class is *pseudo-Anosov* if it is neither periodic nor reducible (it fixes no finite set of disjoint essential curves); see [4, 6]. We write $i(x, y)$ for the geometric intersection number of isotopy classes and T_x for the left Dehn twist about x.

Let f be a mapping class and γ a simple closed curve. Fathi's single-twist theorem [5, Theorem 0.1] says that, on a closed surface, if the f-orbit of γ fills, then $T_\gamma^n f$ is pseudo-Anosov except for at most seven consecutive integers. Ivanov [9, 8, Problem 5] asked for a surface-dependent bound for a multitwist. Let t be a Dehn multitwist $T_{\alpha_1}^{\pm 1} \cdots T_{\alpha_n}^{\pm 1}$ about disjoint curves $\alpha_1, \ldots, \alpha_n$, and suppose that the curves $f^m(\alpha_i)$, $1 \leq i \leq n$, $m \in \mathbb{Z}$, together fill S, that is, every essential curve has nonzero geometric intersection number with one of them. His consecutive-integers question asks whether the j with $t^j \circ f$ not pseudo-Anosov always lie among N_S consecutive integers, for a constant N_S depending only on S. His weaker count question bounds only the number of exceptions. The following examples answer the first question on the five-holed sphere. The curves in both theorems are constructed in Section 3 and shown in Figure 1.

Theorem 1. *Let S be the sphere with five boundary components. There are essential simple closed curves a, d, c, C', C on S such that for every $k \geq 1$ and each $\varepsilon \in \{1, -1\}$ the mapping class $f_k = T_a^k T_d$ and the multitwist $t = T_a T_c^\varepsilon$ satisfy:*

1. *a and c are disjoint and not isotopic, so t is a Dehn multitwist with two components and nonzero exponents (a product of left twists when $\varepsilon = 1$);*

2. *the set* $\{f_k^m(a), f_k^m(c) : m \in \mathbb{Z}\}$ *fills* S*; already* $\{a, c, f_k(a)\}$ *does;*

3. $t^0 f_k = T_a^k T_d$ *fixes* C' *and* $t^{-k} f_k = T_c^{-\varepsilon k} T_d$ *fixes* C*, so neither is pseudo-Anosov.*

Hence the set of j *with* $t^j f_k$ *not pseudo-Anosov contains* 0 *and* $-k$*, and no constant* N_S *as in the consecutive-integers question exists for this* S*.*

Composition is right to left, as in Ivanov's $t^j \circ f$: $T_a^k T_d$ applies T_d first. The theorem holds verbatim for right-handed twists, and for $f_k t^j$ in place of $t^j f_k$, since the two are conjugate by f_k (Remark 5).

Remark 2 (what the example does not show). In the family $t^j f_k$ the exceptional set contains 0 and $-k$; we do not determine it, so the example bears only on where exceptions lie, not on how many there are. The original base maps f_k are reducible. Their a-orbits do not fill individually, since $T_a^k T_d$ fixes C' and $i(a, C') = 0$. The next theorem provides different, pseudo-Anosov bases; its filling proof continues to use the union of component orbits.

Theorem 3. *On the same five-holed sphere, with the same curves and left-twist convention, put*

$$g_k = T_a^{-1} T_c^{-(k+1)} T_d, \qquad t = T_a T_c \qquad (k \geq 1).$$

Then g_k *is pseudo-Anosov, and the union* $\{g_k^m(a), g_k^m(c) : m \in \mathbb{Z}\}$ *fills* S*; already* $\{a, c, g_k(a)\}$ *fills. For every* $n \in \mathbb{Z}$*,*

$$t^n g_k = T_a^{n-1} T_c^{n-k-1} T_d.$$

The member at $n = 1$ *fixes* C*, and the member at* $n = k + 1$ *fixes* C'*. Consequently the exceptional exponents need an interval of at least* $k + 1$ *consecutive integers, even with a pseudo-Anosov base and a fixed positive multitwist. The statement also holds on reversing every twist.*

Long and Morton's single-twist finiteness theorem assumes a pseudo-Anosov base [10, Theorem A]. Fathi's hypothesis allows any base with a filling orbit, as also stated by Cumplido and Wiest [2, Theorem 3]. For a pseudo-Anosov base, Boyer, Gordon and Zhang improved seven to six [1, abstract]; see Watanabe [12, §1.1], who restated Ivanov's question. For a special class of pseudo-Anosov maps (those that become trivial when a chosen puncture is filled in) and multitwists satisfying a non-compressibility condition, Zhang's preprint [14, Theorem 2] claimed that every composition is pseudo-Anosov, so that no exceptions occur; the author withdrew the preprint in January 2008. In a later paper [15, abstract], Zhang showed that for any Dehn twist t_c on a punctured surface there are pseudo-Anosov maps f_m with $f_m \circ t_c^k$ pseudo-Anosov for all k once m is large, and deduced a corollary for multitwists on two curves.

Proof overview. Our contribution is the separation of two exceptional exponents for a fixed multitwist as the base varies. The cancellation of commuting twists gives the two exceptions; the essential point is verifying Ivanov's union-filling hypothesis. Cutting along a, c leaves pairs of pants, so an essential curve disjoint from them is parallel to one of them; two intersection numbers then suffice. The hardest step is this filling verification; the intersection formulas and pants facts are standard. For the stronger example, Penner's construction [11, Theorem 3.1] proves that the base is pseudo-Anosov. The proof below follows these steps. The full wording of Ivanov's question and the historical source details are in the accompanying supplementary material.

The written argument is accompanied by the algebraic formalization and finite checks described in Section 6 and Appendix A.

The proofs, the Lean formalization, the checking programs and the text were produced by AI models (from OpenAI and Anthropic), working under the author's direction. The proof is written, with Lean and computer checks, and has not yet been reviewed by experts.

2 Conventions and standard facts

We specify the surface conventions and recall the intersection formulas used below.

Throughout, S is the compact sphere with five boundary components. Ivanov's Mod_S is $\pi_0(\mathrm{Diff}(S))$, so isotopies may move the boundary. Isotopy classes of curves, the action of mapping classes on them, geometric intersection numbers and the Nielsen–Thurston types are therefore those of the five-punctured sphere $S_{0,5}$, and we work there. A simple closed curve is *essential* if it bounds neither a disk nor a disk with one puncture; on $S_{0,5}$ this means that each side contains at least two of the five punctures. This is Ivanov's *nontrivial* circle, which he describes in his Problem 6 as *not bounding a disk and non deformable into the boundary* [9]. (With a weaker reading the filling hypothesis could never hold, since an inessential curve meets nothing.) Ivanov's Mod_S includes orientation-reversing classes; our example uses orientation-preserving ones only.

We use these standard facts:

(D) If $i(x,y)=0$ then $T_x(y)=y$ and $T_xT_y=T_yT_x$, since the twists have disjoint supports.

(I1) $i(T_x^m(y),y)=|m|\,i(x,y)^2$ for all $m\in\mathbb{Z}$ [4, Prop. 3.2].

(I2) $\bigl|\,i(T_x^m(y),z)-|m|\,i(x,y)\,i(x,z)\bigr|\le i(y,z)$ for all $m\in\mathbb{Z}$ [4, Prop. 3.4].

(N) $i(h(x),h(y))=i(x,y)$ for every mapping class h.

(P) A pseudo-Anosov class fixes no isotopy class of essential curve (Nielsen–Thurston classification [4, Thm. 13.2]; see also [6]).

(Q) A pair of pants (a sphere with three boundary components or punctures) contains no essential curve that is not parallel to a boundary component or puncture. Indeed, a simple closed curve in it separates it, and one of the two sides contains at most one of the three boundary components or punctures, so that side is a disk or an annulus.

3 The curves

Model S as the closed disk D with four punctures p_1,p_2,p_3,p_4 on a horizontal diameter, in this order; the boundary circle ∂D plays the role of the fifth puncture p_5 (collapsing it to a point gives $S_{0,5}$). For a set I of consecutive indices, the *round curve* about I is the boundary of a convex neighborhood of $\{p_i : i\in I\}$ that contains no other puncture. Our five curves (Figure 1) are

$$a=\mathrm{round}\{1,2\},\quad d=\mathrm{round}\{2,3\},\quad c=\mathrm{round}\{3,4\},$$

$$C'=\mathrm{round}\{1,2,3\},\quad C=\mathrm{round}\{2,3,4\}.$$

Each separates the five punctures into sets of sizes 2 and 3, so each is essential; and they induce five different partitions of $\{1,\dots,5\}$, so they are pairwise non-isotopic. Equivalently, with $p_1,\dots,p_5$ placed at the corners of a convex pentagon in this cyclic order, a, d, c, C', C are the boundaries of thin neighborhoods of the sides p_1p_2, p_2p_3, p_3p_4, p_4p_5, p_5p_1.

Lemma 4 (the pentagon pattern)**.** *In the cyclic order (a,d,c,C',C), consecutive curves have geometric intersection number 2 and non-consecutive curves are disjoint:*

$$i(a,c)=i(a,C')=i(d,C')=i(d,C)=i(c,C)=0,$$
$$i(a,d)=i(d,c)=i(c,C')=i(C',C)=i(C,a)=2.$$

Proof. Round curves about runs I, J are disjoint when $I \subseteq J$, $J \subseteq I$ or $I \cap J = \emptyset$: choose nested or disjoint convex neighborhoods. This gives the five zeros. In each of the other five pairs the runs overlap partially, and the round curves can be drawn meeting in two points, as in Figure 1. Each region bounded by one arc of each curve contains a puncture, so there is no bigon, and by the bigon criterion [4, Prop. 1.7] the curves are in minimal position, so the intersection number is 2. □

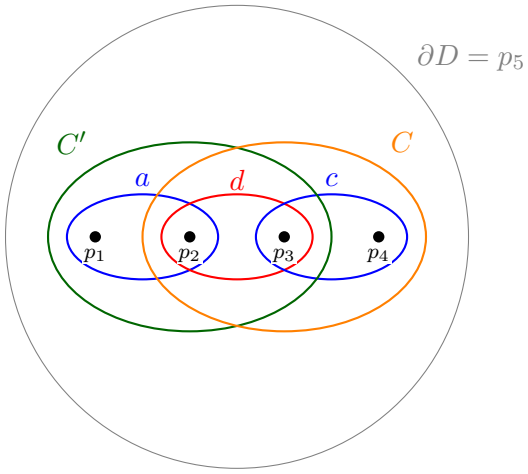


Figure 1: The five curves in the disk model: D with four punctures $p_1, \dots, p_4$ on a diameter, the boundary circle playing the role of the fifth boundary component. Each curve encloses a run of consecutive punctures: a, d, c enclose $\{1,2\}$, $\{2,3\}$, $\{3,4\}$, and C', C enclose $\{1,2,3\}$, $\{2,3,4\}$. In the cyclic order a, d, c, C', C consecutive curves meet twice and the others are disjoint. The multitwist is $t = T_a T_c^{\pm 1}$ (blue curves) and $f_k = T_a^k T_d$.

4 The reducible-base examples

Cancellation gives the exceptional members in Theorem 1; a pants decomposition verifies its filling hypothesis.

Proof of Theorem 1. We prove its three items in turn. Fix $k \geq 1$ and $\varepsilon \in \{1, -1\}$, and put $f = f_k = T_a^k T_d$ and $t = T_a T_c^{\varepsilon}$.

(1) The multitwist. By Lemma 4, a and c are disjoint, and they are not isotopic. Both exponents in t are nonzero, and for $\varepsilon = 1$ the multitwist t is a product of left twists.

(2) The orbits fill. We first show that an essential curve γ with $i(\gamma, a) = i(\gamma, c) = 0$ is isotopic to a or to c. Suppose it is isotopic to neither. Cutting S along the disjoint curves a and c leaves three pairs of pants: one bounded by a, p_1, p_2; one by c, p_3, p_4; and one by a, c, p_5. Realize a, c, γ by geodesics of a complete hyperbolic metric; distinct simple closed geodesics are in minimal position [4, §1.2], so γ is disjoint from $a \cup c$ and lies in one of the three pants. By (Q) it is parallel to a, to c or to a puncture, which contradicts our assumption or the essentiality of γ.

Next we compute two intersection numbers of $f(a) = T_a^k T_d(a)$. Since $i(a, a) = i(a, c) = 0$, the twist T_a^k fixes a and c by (D). By (N) and (I1),

$$i(a, f(a)) = i(T_a^{-k}(a), T_d(a)) = i(a, T_d(a)) = i(d, a)^2 = 4.$$

By (N), $i(c, f(a)) = i(T_a^{-k}(c), T_d(a)) = i(c, T_d(a))$, and (I2) with $x = d$, $y = a$, $z = c$, $m = 1$ gives $\big|\, i(T_d(a), c) - i(d, a)\, i(d, c)\big| \le i(a, c) = 0$, so $i(c, f(a)) = 2 \cdot 2 = 4$.

Now let γ be any essential curve. If γ is disjoint from both a and c, it is a or c, and both meet $f(a)$. So some member of $\{a, c, f(a)\}$ meets γ, and this set, which lies in $A = \{f^m(a), f^m(c) : m \in \mathbb{Z}\}$, fills S.

(3) Two exceptional members, k apart. By (D), T_a and T_c commute, so $t^j = T_a^j T_c^{\varepsilon j}$ and

$$t^j f = T_a^{k+j}\, T_c^{\varepsilon j}\, T_d \qquad (j \in \mathbb{Z}).$$

For $j = 0$ this is $T_a^k T_d$. By Lemma 4, $i(d, C') = i(a, C') = 0$, so T_d and T_a fix C' by (D), and so does $T_a^k T_d$. For $j = -k$ it is $T_c^{-\varepsilon k} T_d$. By Lemma 4, $i(d, C) = i(c, C) = 0$, so T_d and T_c fix C, and so does $T_c^{-\varepsilon k} T_d$. Since C and C' are essential, neither member is pseudo-Anosov, by (P).

(4) Conclusion. For each $k \ge 1$ the hypothesis of Ivanov's question holds for (f_k, t), and the set of j with $t^j f_k$ not pseudo-Anosov contains 0 and $-k$. Any run of consecutive integers that contains both has at least $k + 1$ elements. If N_S existed, taking $k = N_S$ would give a contradiction. So no such N_S exists for the sphere with five boundary components. □

Remark 5 (conventions). With right-handed twists every T_x becomes T_x^{-1}. The proof uses only (D), (I1), (I2) and (N), which hold for both, so it applies verbatim. Since $f\, t^j = f\, (t^j f)\, f^{-1}$, the members $f t^j$ and $t^j f$ are conjugate and have the same exceptional j.

5 Pseudo-Anosov bases

Penner's construction proves the additional assertion in Theorem 3, that the base is pseudo-Anosov.

Proof of Theorem 3. The version of Penner's construction we use is the following [11, Theorem 3.1, p. 187, and definitions on pp. 180–181]. On an oriented finite-type surface of negative Euler characteristic, take two internally disjoint multicurves with essential nonperipheral components, no parallel repetitions, efficient intersections, and union filling by disks and peripheral annuli or punctured disks. Every word in positive right twists on the first family and negative right twists on the second is pseudo-Anosov if every generator occurs. Here peripheral regions are parallel to an end, and efficient intersections exclude unpunctured bigons. We now verify these qualifications. The curves a, c are distinct disjoint essential nonperipheral components, and d is essential and distinct from both. There are no parallel repetitions. Lemma 4 gives $i(a, d) = i(c, d) = 2$ and minimal position, with no unpunctured bigons. The complement of $a \cup d \cup c$ has six disk faces after capping the five ends: four singly punctured bigons, a singly punctured outer quadrilateral, and one unpunctured quadrilateral. Equivalently, on the compact surface its regions are five peripheral annuli and one disk. These are exactly allowed filling regions in Penner's definition [11, p. 181]; efficiency excludes unpunctured bigons [11, p. 180].

Figure 2 shows the three curves; their numerical specification is in Appendix A. The last marking represents the exterior end. These convex neighborhoods are the actual round classes of Section 3: an increasing horizontal homeomorphism identifies the ordered interior markings, and the fifth marking moves within its exterior face. Partitions alone are not used to classify arbitrary curves. Exact crossings and facial walks certify four crossings, eight edges after suppressing ordinary corners, and six Jordan disk faces, with the puncture contents just listed. The finite topology checks of this diagram are recorded in Appendix A.

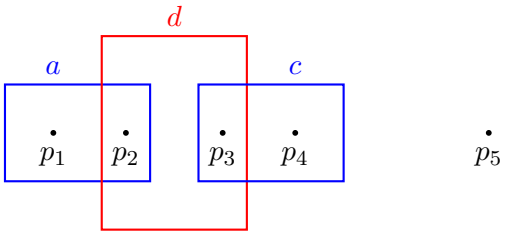


Figure 2: The three curves used in Penner's construction, in the rational coordinates specified in Appendix A. The exterior marking represents the fifth end.

Penner uses right twists τ_x, so our left twists satisfy $T_x = \tau_x^{-1}$. Thus $g_k = \tau_a \tau_c^{k+1} \tau_d^{-1}$ is in his semigroup with positive family $\{a, c\}$ and negative family $\{d\}$. Every generator is used, with multiplicities $1, k+1, 1$, and the oriented surface has Euler characteristic -3. His theorem makes g_k pseudo-Anosov. Global reversal exchanges the two signed families and preserves this conclusion.

For filling, set $M_k = T_a^{-1} T_c^{-(k+1)}$. It fixes a, c by (D). Thus $g_k = M_k T_d$. By (N), (I1), (I2) and Lemma 4,

$$i(a, g_k(a)) = i(a, T_d(a)) = 4, \qquad i(c, g_k(a)) = i(c, T_d(a)) = 4.$$

For the second equality the error in (I2) is at most $i(a, c) = 0$. Cutting along a, c gives the three pants $(p_1, p_2, a), (p_3, p_4, c), (a, c, p_5)$. As in the proof of Theorem 1, an essential curve disjoint from a, c is isotopic to one of them, and hence meets $g_k(a)$. This proves filling by the finite subfamily using only orbit indices $0, 1$.

Disjoint supports give $T_a T_c = T_c T_a$, so combining integer powers proves the displayed normal form for every $n \in \mathbb{Z}$. At $n = 1$ it is $T_c^{-k} T_d$, which fixes the actual C since $i(c, C) = i(d, C) = 0$. At $n = k+1$ it is $T_a^k T_d$, which fixes the actual C' since $i(a, C') = i(d, C') = 0$. Both reducing curves have a $2 \mid 3$ partition and are essential. By (P) both members fail to be pseudo-Anosov. The exponents are k apart, so an interval containing them has at least $k+1$ integers. □

6 Computational verification

We relate the identities formalized in Lean to the surface argument.

The model. The fundamental group of S (the disk with four punctures) is free on loops $x_1, \ldots, x_4$ around $p_1, \ldots, p_4$, with $x_1 x_2 x_3 x_4$ parallel to ∂D. Choosing a basepoint and representatives gives the Artin automorphisms for half twists

$$\sigma_i : \; x_i \mapsto x_i x_{i+1} x_i^{-1}, \quad x_{i+1} \mapsto x_i, \quad x_m \mapsto x_m \; (m \neq i, i+1),$$

and the twists about the round curves are the full twists of the enclosed strands [4, Ch. 9]: $T_a = \sigma_1^2$, $T_d = \sigma_2^2$, $T_c = \sigma_3^2$, $T_{C'} = (\sigma_1 \sigma_2)^3$, $T_C = (\sigma_2 \sigma_3)^3$, up to one global choice of handedness. The canonical fundamental-group action is only defined up to inner automorphism; this ambiguity disappears on conjugacy classes. The five curves are represented, up to conjugacy and inversion, by the words $x_1 x_2$, $x_2 x_3$, $x_3 x_4$, $x_1 x_2 x_3$, $x_2 x_3 x_4$.

The group identities are formalized in Lean. Their interpretation as statements about curves uses the standard surface facts of Section 2. The finite checks are given in Appendix A. Here is the correspondence with Theorem 3. In the following displays, the twist symbols denote the corresponding Artin endomorphisms of F_4. Multiplication denotes composition of endomorphisms,

with the right factor acting first. The formal definitions are $t = T_a T_c$ and $g_k = T_a^{-1} T_c^{-(k+1)} T_d$ for $k \in \mathbb{N}$, where the inverse symbols denote the specified inverse Artin maps. Write $w_C = x_2 x_3 x_4$ and $w_{C'} = x_1 x_2 x_3$ for the words representing C, C'. The formal theorems `Multitwist.V2.fixed_C` and `Multitwist.V2.fixed_Cp` state

$$((t^1)g_k)(w_C) = w_C, \qquad ((t^{k+1})g_k)(w_{C'}) = w_{C'} \quad (k \in \mathbb{N}).$$

For Theorem 1, the formal statements `Multitwist.F3_j0`, `F3_jk_pos`, and `F3_jk_neg` are, in the same notation,

$$(T_a^k T_d)(w_{C'}) = w_{C'},$$
$$((T_a^{-1} T_c^{-1})^k (T_a^k T_d))(w_C) = w_C,$$
$$((T_a^{-1} T_c)^k (T_a^k T_d))(w_C) = w_C \qquad (k \in \mathbb{N}).$$

The last two equations are the two signs of the member at exponent $-k$. The formal theorem `Multitwist.V2.integer_identity` states, for every group G, elements $A, C, D \in G$ with $AC = CA$, and $k, n \in \mathbb{Z}$,

$$(AC)^n (A^{-1} C^{-(k+1)} D) = A^{n-1} C^{n-k-1} D.$$

The concrete fixed-word statements use natural k, including zero; the surface theorems require $k \geq 1$. The integer identity supplies the normal form for all integers n. Further formal nonconjugacy statements show that the words representing $f_k(a)$ and $g_k(a)$ are moved by the twists about a and c, even up to conjugacy and inversion. These are the algebraic counterparts of the positive intersection numbers used in the filling arguments. Both choices of ε occur in the displayed identities for Theorem 1; reversing all twists is justified in Remark 5 and the proof of Theorem 3. The interpretation of Artin maps as twists, the injection of essential curve classes into conjugacy classes up to inversion, the intersection formulas, filling, Penner's theorem and the Nielsen–Thurston implication are supplied by the cited surface results and the topological arguments above. The curve–word correspondence is a standard fact used here, not a claim that it or the surface theory has been formalized. The standard logical axioms and reproduction commands are listed in the checking notes.

Appendix A Finite checks

We give the numerical specifications and routine finite verifications used alongside the main argument.

For the complementary regions, Figure 2 uses punctures $p_1 = (2, 0)$, $p_2 = (5, 0)$, $p_3 = (9, 0)$, $p_4 = (12, 0)$, $p_5 = (20, 0)$ and rectangles with coordinate bounds

$$a = [0, 6] \times [-2, 2], \quad d = [4, 10] \times [-4, 4], \quad c = [8, 14] \times [-2, 2].$$

For the diagram in Figure 2, two independent implementations check the crossings and facial walks; neither infers the complementary regions solely from intersection numbers.

The algebraic model in Section 6 has the following five finite facts:

F0. Each of $T_a, T_d, T_c, T_{C'}, T_C$ fixes its own curve.

F1. The pentagon pattern of Lemma 4: $T_u(v) = v$ for the ten ordered disjoint pairs and $T_u(v) \neq v$ for the ten ordered meeting pairs.

F2. $T_a T_c = T_c T_a$.

F3. For every k and both signs, $T_a^k T_d$ fixes C' and $t^{-k} f_k = T_c^{-\varepsilon k} T_d$ fixes C.

F4. For every k, $T_a(f_k(a)) \neq f_k(a)$ and $T_c(f_k(a)) \neq f_k(a)$, that is, $f_k(a)$ meets a and c.

The free-group calculation proves these facts for every natural k and both signs. It also proves the equalities and nonconjugacy statements for g_k, with the formal statements displayed in Section 6.

The independent curve calculation uses Dynnikov coordinates on the four-punctured disk [3], the update rules of Hall and Yurttaş [7], and the intersection formula of Yurttaş and Hall [13, Lemma 9]. It confirms the pentagon pattern and the exact intersection values $i(a, f_k(a)) = i(c, f_k(a)) = 4$ in finite instances. The supplementary checking notes give the exact programs, ranges, test inputs and logical axioms.

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