Relations among cycle invariants of a three-armed quiver

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Abstract

Join two vertices by three paths, each with at least two edges, and give each edge an arrow in each direction, with variables d_{ij}, u_{ij}. Put $D_i = \prod_j d_{ij}$ and $U_i = \prod_j u_{ij}$, and impose $D_1 - D_2 + D_3 = 0$. Put $v_{ij} = d_{ij}u_{ij}$ and $A = \prod_j v_{1j}$, $B = \prod_j v_{2j}$, $C = \prod_j v_{3j}$. Under independent rescalings at the vertices, the ring of invariant polynomial functions over $\mathbb{C}$ has presentation

$$\mathbb{C}[v_{ij}, w_1, w_2, w_3]/I_2\begin{pmatrix} w_2 & w_3 & B \\ A & w_3 + C & w_1 \end{pmatrix}.$$

Here $(w_1, w_2, w_3) = (D_1U_2, D_2U_1, -D_2U_3)$, and I_2 denotes the ideal of 2×2 minors. Makonzi proved that these elements generate the invariant ring. We prove that the minors generate all their relations, with the corrected all-edge indexing of Remark 3, establishing the kernel formulation of Makonzi's conjecture. The key step is a specialization to a rational function field that proves independence of normal forms. Lean checks the kernel equality over $\mathbb{C}$ for every triple of path lengths at least two.

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1 Introduction

We prove the determinantal kernel presentation in Makonzi's Conjecture 5.2 for a doubled three-armed quiver [1]. Fix integers $p_1, p_2, p_3 \geq 2$. For $i = 1, 2, 3$ and $1 \leq j \leq p_i$, let d_{ij}, u_{ij}, v_{ij} be indeterminates and put

$$D_i = \prod_j d_{ij}, \qquad U_i = \prod_j u_{ij}, \qquad P = \mathbb{C}[v_{ij}], \qquad H = \mathbb{C}[d_{ij}, u_{ij}]/(D_1 - D_2 + D_3).$$

We also write D_i, U_i for their images in H. Set $S = P[w_1, w_2, w_3]$, and define the $\mathbb{C}$-algebra map $\Phi : S \to H$ by

$$(w_1, w_2, w_3) \longmapsto (D_1U_2, D_2U_1, -D_2U_3), \qquad v_{ij} \longmapsto d_{ij}u_{ij}.$$

Write $A = \prod_j v_{1j}$, $B = \prod_j v_{2j}$, $C = \prod_j v_{3j}$.

Theorem 1 (Makonzi's Conjecture 5.2)**.** *With the all-edge indexing $1 \leq j \leq p_i$ of Remark 3, the kernel of Φ is the ideal J generated by the 2×2 minors of*

$$\begin{pmatrix} w_2 & w_3 & B \\ A & w_3 + C & w_1 \end{pmatrix}.$$

The new step is the kernel equality; generation and vanishing of the minors are Makonzi's results. We first reduce polynomials modulo J to P-linear combinations of 1 and powers of a single w_i. The key choice is the specialization $\psi : H \to \mathrm{Frac}(P)(t)$ in the proof: it sends w_3 to t, while the images of w_1, w_2 have poles at $0, -C$, respectively. Comparing these poles proves independence of the normal forms and hence $\ker \Phi \subseteq J$.

In Makonzi's quiver construction, take two vertices s, q joined by three paths (the arms), the ith having p_i edges and $p_i - 1$ interior vertices. Double each edge: d_{ij} points from s towards q along the path, and u_{ij} points back. Then D_i, U_i are path products and $d_{ij}u_{ij}$ and D_iU_k are cycles. Figure 1 depicts these paths schematically. We take $p_i \geq 2$, corresponding to three nonempty arms in the dual graph of [1, §2], where arm i has $p_i - 1$ vertices; the algebraic role of this assumption is discussed in Remark 7. The ring H is the quotient defined in [1, Eq. (3.1)].

Let G be the torus with one copy of $\mathbb{C}^*$ for each vertex of the quiver. For $\mu = (\mu_v)_v \in G$ and an arrow a with tail $\mathrm{tail}(a)$ and head $\mathrm{head}(a)$, use Makonzi's action

$$\mu \cdot a = \mu_{\mathrm{tail}(a)}^{-1} a \mu_{\mathrm{head}(a)}.$$

Each D_i has weight $\mu_s^{-1}\mu_q$, so the defining ideal of H is stable and the action descends to H. We take the following generation result from Makonzi's Lemma 4.1, Proposition 4.2, Lemma 4.3 and Proposition 5.1 [1]:

$$H^G = \mathbb{C}[d_{ij}u_{ij}, D_1U_2, D_2U_1, -D_2U_3].$$

Proposition 5.1 also proves that the three minors vanish. For completeness, an invariant class in H has a weight-zero polynomial lift: decompose any lift into torus weights and discard the nonzero weights, since the quotient map preserves weights. A weight-zero monomial has equal incoming and outgoing multiplicities at every vertex. The associated directed multigraph decomposes into directed cycles, by successively following arrows until a vertex repeats and removing that cycle. Thus cycle monomials generate the invariants. Makonzi's Proposition 4.2 and Lemma 4.3 reduce these to the displayed finite list. Write $\theta : S \to H^G$ for this surjective map and $\iota : H^G \hookrightarrow H$ for inclusion. Then $\Phi = \iota \circ \theta$, so $\ker \Phi = \ker \theta$. Conjecture 5.2 asserts precisely that $\ker \theta$ is the ideal generated by the 2×2 minors of that matrix. Theorem 1 therefore gives:

Corollary 2. *The map θ induces an isomorphism $S/J \cong H^G$.*

Remark 3 (Source conventions). We use the published kernel formulation. ArXiv v1 instead states: "The ring homomorphism φ is an isomorphism" [3, Conjecture 5.2]. That map is unquotiented; the nonzero minor relations prevent its injectivity. ArXiv v2 has the kernel formulation [2]. The published equations (3.1) and (5.1) are (3.A) and (5.A) in v2. The map printed in §5.1 uses $1 < j \leq p_i$; we use $1 \leq j \leq p_i$, as required by the first-edge entries in the matrix and §4.3.

Setting all $v_{ij} = v$ in the matrix, hence $(A, B, C) = (v^{p_1}, v^{p_2}, v^{p_3})$, gives the rational determinantal surface singularity of [1, §1.2], whose minimal resolution has a star-shaped dual graph. The conjecture concerns the ring before these identifications.

Bruns and Vetter's Theorems (2.7) and (2.16) give perfection and resolutions for maximal-minor ideals under a grade hypothesis; their Theorem (16.36) gives the Hilbert–Burch description when a quotient has a two-step free resolution [6]. Our normal forms instead give a P-basis of S/J and embed it in a field, proving both the kernel equality and the domain assertion.

Immediately after Conjecture 5.2, Makonzi writes that calculations in Magma "confirm this in small cases, but the Gröbner approach seems hard in general" [1, p. 17, paragraph following Conjecture 5.2]. An analogous quasideterminantal presentation for reconstruction algebras of type A (cyclic quotient singularities) is Makonzi's Theorem 1.1 [4]. Reconstruction algebras and their

resolution interpretation were developed by Wemyss [7, Definition 2.3, Theorems 3.24 and 4.5]; the star-shaped algebra in [1, §2] follows the double-quiver presentation of Iyama and Wemyss [5, Theorem 5.7]. Wahl relates determinantal rational surface singularities to $2 \times (e-1)$ matrices and their exceptional curves [8, Proposition 3.2 and Theorem 3.4]. Craw and Yamagishi give general trace generators and relation ideals for quiver invariant rings [9, Theorems 1.1 and 4.3]. Their relation ideal lies in the invariant ring of the unconstrained quiver. Here we give the three defining relations in Makonzi's specified polynomial generators.

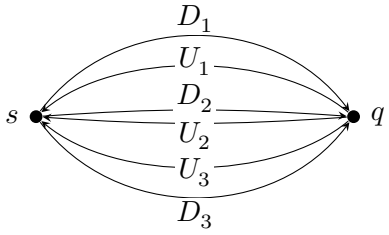


Figure 1: Each pair of curves abbreviates a path of p_i edges and its reverse. The $p_i - 1$ interior vertices of each path are suppressed; D_i, U_i denote products along the entire paths.

This work has not yet been reviewed by experts. The proofs, the Lean formalization and the text were produced by AI models (from OpenAI and Anthropic), working under the author's direction.

2 Proof of Theorem 1

The three minors give, modulo J,

$$w_1 w_2 = AB, \qquad w_1 w_3 = B w_3 + BC, \qquad w_2 w_3 = A w_3 - C w_2. \tag{1}$$

We first find normal forms over P, then prove their uniqueness after mapping H to a rational function field. This gives the inclusion $\ker \Phi \subseteq J$.

Lemma 4. *S/J is generated as a P-module by $1, w_1^n, w_2^n, w_3^n$ for $n \geq 1$.*

Proof. For a monomial involving at least two of w_1, w_2, w_3, apply the relevant identity in (1). Each resulting monomial has smaller total degree in w_1, w_2, w_3. Degree induction gives elements of P or powers of a single w_i; linearity gives the assertion, with coefficients in P. □

Lemma 5. *For a field K, an indeterminate t and nonzero $\alpha, \beta, \gamma \in K$, the elements*

$$1, \quad \left(\frac{\beta(t+\gamma)}{t}\right)^n, \quad \left(\frac{\alpha t}{t+\gamma}\right)^n, \quad t^n \qquad (n \geq 1)$$

of the rational function field $K(t)$ are linearly independent over K.

Proof. If the family $(\beta(t+\gamma)/t)^n$ has a nonzero coefficient in a finite relation, let $m \geq 1$ be the largest exponent whose coefficient a_m is nonzero. After multiplying by t^m, all terms are regular at $t = 0$ since $\gamma \neq 0$. All vanish there except the mth power of the first family, which contributes $a_m(\beta\gamma)^m \neq 0$, a contradiction. Thus every coefficient in that family is zero. Apply the same argument to $(\alpha t/(t+\gamma))^n$: if b_m is its highest nonzero coefficient, multiplication by $(t+\gamma)^m$ and evaluation at $t = -\gamma$ leave only $b_m(-\alpha\gamma)^m \neq 0$, again a contradiction. The remaining relation is a polynomial in t; all its coefficients vanish. □

Proof of Theorem 1. The relation $D_1 - D_2 + D_3 = 0$ gives $\Phi(w_3 + C) = -D_1 U_3$, so the three minors vanish under Φ. Hence $J \subseteq \ker \Phi$, as also proved in [1, Proposition 5.1]. For the reverse inclusion, take $K = \mathrm{Frac}(P)$, with the v_{ij} algebraically independent, and let t be transcendental over K. Thus A, B, C are nonzero in K.

Define a $\mathbb{C}$-algebra homomorphism from $\mathbb{C}[d_{ij}, u_{ij}]$ to $K(t)$ by

$$d_{11} \mapsto (t + C)/t, \qquad d_{21} \mapsto 1, \qquad d_{31} \mapsto -C/t,$$

send the other d_{ij} to 1, and send u_{ij} to v_{ij} divided by the image of d_{ij}. Each image of d_{ij} is nonzero. The relation $D_1 - D_2 + D_3$ maps to $(t+C)/t - 1 - C/t = 0$, so the map descends to $\psi : H \to K(t)$. The images of (U_1, U_2, U_3) are $(At/(t + C), B, -t)$. Thus $\psi \circ \Phi$ satisfies

$$v_{ij} \mapsto v_{ij}, \qquad (w_1, w_2, w_3) \mapsto \left(\frac{B(t + C)}{t}, \frac{At}{t + C}, t \right).$$

It suffices to show $\ker(\psi \circ \Phi) \subseteq J$, since $\ker \Phi \subseteq \ker(\psi \circ \Phi)$. For $f \in \ker(\psi \circ \Phi)$, Lemma 4 gives $f - h \in J$ with

$$h = r + \sum_{n=1}^{N} (a_n w_1^n + b_n w_2^n + c_n w_3^n), \qquad r, a_n, b_n, c_n \in P.$$

Since $J \subseteq \ker \Phi$, we have $\psi\Phi(h) = 0$. Apply Lemma 5 with $(\alpha, \beta, \gamma) = (A, B, C)$ and the indeterminate t. Every coefficient vanishes in K, hence in P by the injection $P \hookrightarrow K$. Thus $h = 0$ and $f \in J$. Since $\psi\Phi(J) = 0$, we have $\ker(\psi \circ \Phi) = J$ and hence $\ker \Phi = J$. □

Corollary 6. *The ring S/J is a domain and a free P-module with basis $1, w_1^n, w_2^n, w_3^n$ for $n \geq 1$.*

Proof. The map $\psi\Phi$ embeds S/J into $K(t)$. Lemma 4 gives spanning over P, and Lemma 5 gives independence. □

In particular, $\mathrm{Spec}(S/J) \to \mathrm{Spec}\, P$ is flat, since S/J is a free P-module.

Remark 7. For the map Φ defined above, the same kernel equality holds over any field k for $p_i \geq 1$. Indeed, with $P = k[v_{ij}]$, the monomials A, B, C are still nonzero, and Lemmas 4 and 5 apply in every characteristic. The specialization assigns special values to d_{11}, d_{21}, d_{31} and sends each d_{ij} with $j > 1$ to 1; it sends every u_{ij} to v_{ij} divided by the chosen image of d_{ij}. Thus all v_{ij} are retained, and only the existence of the first edge of each path is needed. The invariant-ring interpretation above uses Makonzi's three nonempty arms, with $p_i \geq 2$.

3 Formal verification

The main formal theorem is `Makonzi.kernel_equality`:

```
theorem kernel_equality (p : Fin 3 → ℕ) (hp : ∀ i, 2 ≤ p i) :
    RingHom.ker (phi p) = minorIdeal p
```

The coefficient field in these definitions is $\mathbb{C}$. The source of `phi` is the polynomial ring S, and its target is the arrow polynomial ring modulo $D_1 - D_2 + D_3$, namely H. Its variable images are $v_{ij} \mapsto d_{ij} u_{ij}$ and $(w_1, w_2, w_3) \mapsto (D_1 U_2, D_2 U_1, -D_2 U_3)$. The definition `minorIdeal` is the ideal of the three minors of the matrix in Theorem 1, including its $w_3 + C$ entry. The arm and edge indices are shifted by one in Lean. Thus the displayed equality is exactly $\ker \Phi = J$ for $p_i \geq 2$. Lean [10], with Mathlib [11], checks both containments, normal-form reduction and rational-function

independence. Makonzi's Lemma 4.1, Proposition 4.2, Lemma 4.3 and Proposition 5.1 supply the generation result used in Corollary 2. Corollary 6 and Remark 7 are not formalized. The formalization and its checking guide, including the pinned versions and axiom dependencies, are available at [12].

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