Locked tromino tilings of rectangles

P. Graham
https://orcid.org/0009-0007-8493-1505

October 28, 2026

Abstract

An $m \times n$ rectangle has a locked tromino tiling if and only if $3 \mid mn$ and $\min(m, n) \geq 4$. A tromino consists of three grid cells connected through shared sides; a tiling is locked if no pair of tiles can be recombined into a different pair and the rectangle has another tiling. We prove the existence direction of Tucker-Foltz's conjecture. Tucker-Foltz showed that rectangles with a side of length at most three have none, constructed locked tilings when one side is divisible by three and at least six and the other is at least one hundred, and conjectured that four suffices. Following Tucker-Foltz's construction by compatible rectangular blocks, we supply the remaining cases and glue them along full rows and columns. Aanjaneya and Pal earlier proved existence on the same rectangles for the different condition of fault-free L-tromino tilings. Lean formalizes the existence theorem for $3 \mid m$, $m \geq 6$, $n \geq 4$, including the gluing argument, arithmetic and finite blocks, using only `propext`, `Classical.choice` and `Quot.sound`. Two separately written programs, sharing no code, check the blocks and seams. The classification uses Tucker-Foltz's preprint theorem for narrow rectangles, whose proof leaves the $m = 3$ frontier check to the reader there; we neither repeat nor formalize that check.

1 Introduction

Cut a rectangular grid of cells into *trominoes*, sets of three cells connected through shared sides: straight ones and L-shaped ones, in any orientation. Pick two of the trominoes, remove them, and try to cover the six freed cells with two trominoes in a different way. If this is possible the change is a *recombination move*. A tiling is *locked* if no recombination move is possible, for any two of its tiles, although the grid does have other tromino tilings (Figure 1). We determine exactly which rectangles have a locked tromino tiling.

Theorem 1. *Let m and n be positive integers. The $m \times n$ rectangle has a locked tromino tiling if and only if 3 divides mn and $\min(m, n) \geq 4$.*

Tucker-Foltz introduced locked tilings as obstructions to the ReCom Markov chain of DeFord, Duchin and Solomon [4], which merges two districts and splits their union into two new ones [8, Section 1]. On a grid with equal-population cells and three cells per district, plans are tromino tilings and recombination moves are ReCom steps. Tucker-Foltz proved the necessary narrow-grid exclusion [8, Theorem 3.1] and existence for $3 \mid m$, $m \geq 6$, $n \geq 100$ [8, Theorem 3.2], conjecturing that $n \geq 4$ suffices [8, Section 3].

Theorem 2 (conjectured by Tucker-Foltz)**.** *Let m and n be integers with $3 \mid m$, $m \geq 6$ and $n \geq 4$. Then the $m \times n$ rectangle has a locked tromino tiling.*

The block method is Tucker-Foltz's: the blocks in [8, Figure 7 and proof of Theorem 3.2] cover the family $m = 6u + 9v$, $n = 22x + 19y + 8z$, where $u, v, y, z \geq 0$, $u + v \geq 1$ and $x \geq 1$, and its transposes; in particular they cover $3 \mid m$, $m \geq 6$, $n \geq 100$. Tucker-Foltz also reports locked tilings of the 6×4 rectangle [8, Section 3]; Tucker-Foltz's 6×6 example extends by flipped copies to every $6r \times 6s$, $r, s \geq 1$ [8, Section 1, Figures 2–3]. Our blocks cover all the rectangles in Theorem 2, including the remaining cases. We use compatible blocks as in Tucker-Foltz's construction. The difficult step is finding blocks at narrow widths that remain rigid against one another in both orders and against copies of themselves. Section 3 explains how finitely many such blocks suffice. The existence proof, including their compatibility, is formalized in Lean 4 [7].

Up to transposition, the smallest permitted rectangles by area are 6×4, 6×5, 6×6 and 9×4; only the 6×4 and 6×6 cases among these four were settled before [8, Sections 1 and 3]. This classification concerns the grid model with three-cell districts. Related irreducibility results include Cannon's theorem for three simply connected districts in a triangular region of the triangular lattice, allowing district sizes to vary by at most one vertex [3]. Akitaya, Dituro, Gonczi, Korman, Souvaine, Stock and Tóth simplify the proof, weaken Cannon's hypotheses, and prove a square-grid analogue [2]; both allow size tolerance, as discussed in [5, Section 1.2]. Emre, Jerison and Veomett prove connectivity for a $3 \times n$ grid with three districts of size n [5, Theorem 3].

Aanjaneya and Pal [1, Theorem 2] proved that every rectangle with $m, n \geq 4$ and $3 \mid mn$ admits a fault-free L-tromino tiling, also by extending finitely many small rectangles. A fault line is a full horizontal or vertical cut that crosses no tile; fault-free means no such cut exists. Locked tilings instead forbid two-tile recombinations and allow straight and L-trominoes. These are different conditions: tilings assembled from multiple blocks have fault lines along their joins, while lockedness concerns two-tile recombinations.

The proofs, the Lean formalization, the certificate checkers and the text were produced by AI models (from OpenAI and Anthropic), working under the author's direction. This work has not yet been reviewed by experts.

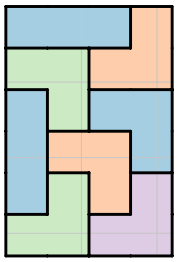


Figure 1: A locked tromino tiling of the 6×4 rectangle. Thick lines are tile boundaries; colors only separate neighboring tiles. It is the height-6 block of width 4 used in the proof.

2 Tilings and rigidity

A *cell* is a pair of integers (i, j), read as row i and column j. Two cells are *adjacent* if they agree in one coordinate and differ by one in the other. The $m \times n$ *rectangle* is the set of cells (i, j) with $0 \leq i < m$ and $0 \leq j < n$; it has m rows and n columns. A *tromino* is a set of three cells that is connected: any two of its cells are joined by a sequence of adjacent cells inside it. Up to translation there are six trominoes, two straight and four L-shaped. A *tiling* of a set of cells is a partition of it into trominoes, called its *tiles*.

Definition 3. Let T be a tiling and A, B two of its tiles. A *recombination move* on $\{A, B\}$ replaces them by two disjoint trominoes C, D with $C \cup D = A \cup B$ and $\{C, D\} \neq \{A, B\}$. The pair $\{A, B\}$ is *rigid* if it admits no recombination move. The tiling T of a rectangle is *locked* if every pair of its tiles is rigid and the rectangle has a tiling other than T.

Tucker-Foltz [8, Section 1] works with the *metagraph* $M(G, 3)$ of a graph G: its vertices are the partitions of G into connected subgraphs of three vertices each, and two partitions are adjacent when they differ only on two parts. A locked tiling is an isolated vertex of $M(G, 3)$ that is not its only vertex. For the grid graph G of a rectangle, the vertices are the tromino tilings, and two tilings that differ only on two parts cover the same six cells with those parts, so its edges are exactly the recombination moves. The two definitions of a locked tiling therefore agree.

Translating a tiling, or reflecting it in a horizontal, vertical or diagonal line, preserves adjacency, so it maps trominoes to trominoes, recombination moves to recombination moves and locked tilings to locked tilings. In particular the transpose $(i, j) \mapsto (j, i)$ turns locked tilings of the $m \times n$ rectangle into locked tilings of the $n \times m$ rectangle.

Lemma 4. *Let A and B be disjoint trominoes such that no cell of A is adjacent to a cell of B. Then $\{A, B\}$ is rigid.*

Proof. Let C and D be disjoint trominoes with $C \cup D = A \cup B$. A path of adjacent cells inside $A \cup B$ that starts in A never reaches B, because its first step into B would join a cell of A to an adjacent cell of B. Since C is connected, it therefore lies inside A or inside B, and since all three sets have three cells, C equals A or B. The same holds for D, and $C \neq D$ because they are disjoint and nonempty. Hence $\{C, D\} = \{A, B\}$. □

Lemma 5. *If m and n are positive integers with $3 \mid m$ and $n \geq 3$, the $m \times n$ rectangle has at least two tromino tilings. Hence a tiling of it is locked as soon as every pair of its tiles is rigid.*

Proof. Cover every column by vertical straight trominoes; or cover the first three columns by horizontal straight trominoes, one in each row, and the remaining columns by vertical ones (Figure 5). The second tiling contains the tile $\{(0, 0), (0, 1), (0, 2)\}$ and the first does not. Any tiling differs from at least one of the two. □

The necessary condition uses Tucker-Foltz's narrow-grid theorem, stated in that paper's notation: $G(m, n)$ is the $m \times n$ grid graph.

Theorem 6 (Tucker-Foltz [8, Theorem 3.1])**.** *For any positive integers m and n such that $m \leq 3$, $M(G(m, n), 3)$ is connected.*

For $m = 3$, Tucker-Foltz's proof reduces to an exhaustive list of frontier configurations (their Figure 6), whose verification is left to the reader there; we neither repeat it nor formalize it.

3 Proof overview

By Lemma 5, it suffices to construct a tiling with every pair rigid. We cut along full rows and columns into rectangular blocks. The gluing lemma shows that recombination can occur only within a block or across a shared side, called a seam. It therefore suffices to choose rigid blocks whose seams are rigid for both orders, including repeated blocks (the directional conventions are defined in Section 4).

Heights 6 and 9 generate every multiple of three from six onward. At each width from 4 to 55, one column of blocks suffices, except that width 5 uses heights 9 and 12 and separate tilings at

heights 6 and 15, which are not sums of nines and twelves. For widths at least 56, sums of eights and nines allow a grid built from four block types with heights 6 and 9 and widths 8 and 9. We prove the gluing assertion, state the finite construction, and then assemble the tilings using these arithmetic facts.

4 Gluing rectangular blocks

Let $m = h_1 + \cdots + h_r$ and $n = w_1 + \cdots + w_s$ with positive integers h_k and w_l. Put $H_k = h_1 + \cdots + h_k$ and $W_l = w_1 + \cdots + w_l$, with $H_0 = W_0 = 0$. The *block* (k, l) is the set of cells (i, j) with $H_{k-1} \le i < H_k$ and $W_{l-1} \le j < W_l$, an $h_k \times w_l$ rectangle. The blocks partition the $m \times n$ rectangle. Two blocks are *side neighbors* if they lie in the same row of blocks and in consecutive columns of blocks, or in the same column of blocks and in consecutive rows of blocks.

Lemma 7 (gluing)**.** *For each block (k, l) let T_{kl} be a tiling of it, and let T be the union of these tilings, a tiling of the $m \times n$ rectangle. Suppose that*

(a) every pair of tiles of each T_{kl} is rigid, and

(b) for every two side neighbors (k, l) and (k', l'), every pair consisting of a tile of T_{kl} and a tile of $T_{k'l'}$ is rigid.

Then every pair of tiles of T is rigid.

Proof. Let A and B be tiles of T, in blocks (k, l) and (k', l'). If the blocks are equal or side neighbors, $\{A, B\}$ is rigid by (a) or (b). Otherwise we show that no cell of A is adjacent to a cell of B, so that Lemma 4 applies. Let $(i, j) \in A$ and $(i', j') \in B$ be adjacent. Then $|i - i'| \le 1$, and since every height h_k is positive, the rows i and i' lie in the same row of blocks or in consecutive ones; the same holds for the columns j and j'. Adjacent cells also agree in one coordinate, so the two blocks share a row of blocks or a column of blocks. Hence they are equal or side neighbors, contrary to assumption. □

Condition (b) concerns only two blocks at a time, and rigidity is unchanged by translation, so it can be checked on a fixed pair of rectangles. For a tiling X of an $h \times w$ rectangle and a tiling Y of an $h' \times w$ rectangle, say that X *stacks on* Y if, after Y is moved down by h rows, every tile of X forms a rigid pair with every tile of the moved Y; the line between them is the seam. Similarly, for equal heights, X *fits left of* Y if this holds after Y is moved right by the width of X. We also call a rectangular tiling a block when its tile partition is understood. A tiling or block is *rigid* if all pairs of its tiles are. Figure 2 shows the four stacks of the two width-4 blocks, and Figure 3 the four blocks for large widths, with both kinds of seam.

Corollary 8. *(i) Let S be a set of rigid blocks of a common width w such that every $X \in S$ stacks on every $Y \in S$, including $Y = X$. Then any column of blocks from S, placed one under another, is a tiling of an $h \times w$ rectangle all of whose pairs of tiles are rigid, where h is the sum of the heights.*

(ii) Let $P_{h,w}$, for h in a set $\mathcal{H}$ and w in a set $\mathcal{W}$, be rigid blocks of size $h \times w$ such that $P_{h,w}$ stacks on $P_{h',w}$ and fits left of $P_{h,w'}$ for all $h, h' \in \mathcal{H}$ and $w, w' \in \mathcal{W}$. If $m = h_1 + \cdots + h_r$ with every $h_k \in \mathcal{H}$ and $n = w_1 + \cdots + w_s$ with every $w_l \in \mathcal{W}$, the tiling that places a translate of P_{h_k,w_l} in block (k, l) has only rigid pairs.

Proof. Both follow from Lemma 7. In (ii), condition (a) is the rigidity of the blocks, and for side neighbors (k,l) above $(k+1,l)$ or left of $(k,l+1)$, condition (b) is the statement that P_{h_k,w_l} stacks on P_{h_{k+1},w_l} or fits left of $P_{h_k,w_{l+1}}$, translated into place. Part (i) is the case of a single column of blocks. □

5 The block construction

The blocks were found by a SAT encoding of tile placements and forbidden recombinations. Their coordinates are in the `data/` folder released with this paper (Appendix A), archived at `https://doi.org/10.5281/zenodo.23638798` [6]. For each block, the finite verification establishes that the cells are partitioned into connected triples. It then enumerates every partition of the six cells of a pair of tiles into two connected triples, checking that only the original pair occurs. This is done inside each block and across every listed seam; pairs with no adjacent cells are rigid by Lemma 4.

Proposition 9. *The following tilings exist and are given explicitly in the `data/` folder (Appendix A).*

(i) For each width w with $4 \le w \le 55$ and $w \ne 5$, rigid blocks $B_{6,w}$ and $B_{9,w}$ of sizes $6 \times w$ and $9 \times w$, each of which stacks on each, including itself.

(ii) Rigid blocks $B_{9,5}$ and $B_{12,5}$ of sizes 9×5 and 12×5, each of which stacks on each, including itself; and rigid tilings of the 6×5 and 15×5 rectangles.

(iii) Rigid blocks $P_{h,w}$ of size $h \times w$ for $h \in \{6,9\}$ and $w \in \{8,9\}$, such that $P_{h,w}$ stacks on $P_{h',w}$ and fits left of $P_{h,w'}$ for all $h,h' \in \{6,9\}$ and $w,w' \in \{8,9\}$.

The width-5 construction uses heights 9 and 12; no nonexistence claim for a compatible height-6 block is needed or established here.

Figures 2, 3 and 6 show the width-4 blocks, the four blocks for large widths and the width-5 tilings.

Proof. Each statement is a finite check of explicit tilings: that each block is a tiling of its rectangle by trominoes, and that each of the listed pairs of tiles, inside a block or across a seam, admits only its own partition into two trominoes. Lean's kernel checks all of them, and so do two separately written programs (Section 7). □

Lemma 10. *(a) Every multiple of 3 that is at least 6 is $6a+9b$ for some integers $a,b \ge 0$.*

(b) Every integer that is at least 56 is $8a+9b$ for some integers $a,b \ge 0$.

(c) Every multiple of 3 that is at least 9, other than 15, is $9a+12b$ for some integers $a,b \ge 0$.

Proof. (a) Write $m = 3q$ with $q \ge 2$. If q is even, $m = 6 \cdot (q/2)$; if q is odd, then $q \ge 3$ and $m = 9+6\cdot(q-3)/2$. (b) Write $n = 8Q+r$ with $0 \le r \le 7$. Then $Q \ge 7 \ge r$ and $n = 8(Q-r)+9r$. (c) Dividing by 3, we need every integer $q \ge 3$ other than 5 to be $3a+4b$: this holds for $q = 3,4,6,7,8$, and adding threes reaches every larger integer. □

Proof of Theorem 2. Let $3 \mid m$, $m \ge 6$ and $n \ge 4$. By Lemma 5 it suffices to find a tiling of the $m \times n$ rectangle all of whose pairs of tiles are rigid.

If $n \ge 56$, Lemma 10(a),(b) writes m as a sum of sixes and nines and n as a sum of eights and nines, and Corollary 8(ii) with the blocks of Proposition 9(iii) gives the tiling.

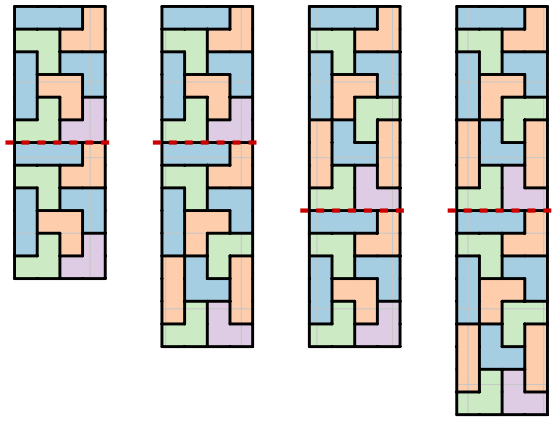


Figure 2: The two blocks of width 4, of heights 6 and 9, stacked in all four orders: 6 on 6, 6 on 9, 9 on 6 and 9 on 9. Each dashed red line is a seam. No two tiles recombine, inside a block or across its seam, so any column of these blocks is a locked tiling.

If $4 \le n \le 55$ and $n \ne 5$, Lemma 10(a) writes m as a sum of sixes and nines, and Corollary 8(i) with the blocks $B_{6,n}$ and $B_{9,n}$ of Proposition 9(i) gives the tiling.

If $n = 5$ and $m \in \{6, 15\}$, Proposition 9(ii) gives the tiling directly. Otherwise $m \ge 9$ and $m \ne 15$, Lemma 10(c) writes m as a sum of nines and twelves, and Corollary 8(i) with $B_{9,5}$ and $B_{12,5}$ gives the tiling. □

6 Proof of the classification

Proof of Theorem 1. Suppose first that the $m \times n$ rectangle has a locked tiling T. Its mn cells are split into tiles of three cells, so $3 \mid mn$. Suppose that $\min(m, n) \le 3$. Transposing if necessary, we may assume $m \le 3$; then $M(G(m, n), 3)$ is connected by Theorem 6. But T is an isolated vertex of this graph, and the graph has another vertex, so it is not connected. This contradiction shows $\min(m, n) \ge 4$.

Conversely, suppose $3 \mid mn$ and $\min(m, n) \ge 4$. Since 3 is prime, it divides m or n; transposing if necessary, $3 \mid m$. As $m \ge 4$ and m is a multiple of 3, $m \ge 6$. Also $n \ge 4$, so Theorem 2 gives a locked tiling of the $m \times n$ rectangle, and transposing back gives one of the original rectangle. □

7 Formal verification

The formal theorem is `LockedTromino.Witnesses.locked_tromino_rectangles`:

$$\forall\, m\ n,\quad 6 \le m \ \to\ 3 \mid m \ \to\ 4 \le n \ \to\ \exists T,\ \mathrm{Locked}\,(\mathrm{Rectangle}\ m\ n)\ T.$$

Here m, n are natural numbers. The formal definitions use integer-pair cells and adjacency through shared sides. The rectangle consists of $0 \le i < m$, $0 \le j < n$; a tile is a connected set of three distinct cells, a tiling is a partition into such sets, and rigidity means that every split of two tiles into two trominoes gives the original unordered pair. The predicate Locked combines this partition and pairwise rigidity with the existence of a different tiling, exactly as in Section 2. Thus the statement is Theorem 2, whose hypotheses make its integer parameters natural numbers, with no

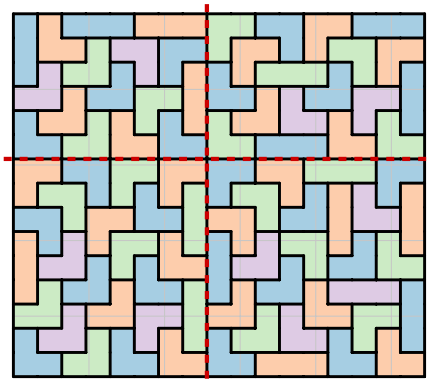


Figure 3: The four blocks for widths $n \geq 56$, of heights 6 and 9 and widths 8 and 9, here in a 15×17 rectangle with the horizontal and vertical seams dashed. Each block also stacks on and fits beside every block of the same width or height, including itself, so these four fill every rectangle with $3 \mid m$, $m \geq 6$ and $n \geq 56$.

additional assumptions. The proof formalizes Lemmas 4, 5, 7 and 10, Corollary 8 and Proposition 9. Theorem 1 additionally uses the written deduction in Section 6 and Tucker-Foltz's Theorem 3.1, which remains outside Lean.

Lean's kernel checks the finite assertions by reduction, using `decide` and `rfl`, without `native_decide` or `Lean.ofReduceBool`. The main existence theorem depends on exactly `propext`, `Classical.choice` and `Quot.sound`. Two separately written programs, sharing no code, independently enumerate all partitions of each six-cell union into two trominoes. The small-count comparison is in Appendix B. Checking notes supplied with the paper give the logical axioms, reproduction commands and resource estimates.

Acknowledgments

Jamie Tucker-Foltz posed the problem, gave the definitions used here, proved the case of narrow grids and the case $n \geq 100$ by the block method we follow, and computed the small cases used to check our enumeration. Lean and its contributors supplied the formal-proof system.

Appendix A The blocks

This appendix describes where the tilings of Proposition 9 are: their coordinates are in the folder `data/` released with this paper, one JSON file per family, and Figures 1, 2, 3 and 6 draw ten of them. Each file names its blocks, gives each block's height, width and tiles (each tile a list of three cells (i, j) with i the row), and lists the seams to be checked: `vertical` $[X, Y]$ means X stacks on Y, and `horizontal` $[X, Y]$ means X fits left of Y.

File in `data/`	Contents	Blocks	Stacks	Side by side
`witnesses/width-w.json` ($w \neq 5$)	$B_{6,w}$, $B_{9,w}$, $4 \leq w \leq 55$	2 each	4 each	–
`witnesses/width-5.json`	$B_{9,5}$, $B_{12,5}$	2	4	–
`witnesses/direct-6x5.json`	a rigid 6×5 tiling	1	–	–
`witnesses/direct-15x5.json`	a rigid 15×5 tiling	1	–	–
`palette.json`	$P_{h,w}$, $h \in \{6, 9\}$, $w \in \{8, 9\}$	4	8	8
Total, 55 files		110	216	8

In all, the proof uses 110 blocks and 224 ordered seams. The seams of each family include both orders of two different blocks and each block against a copy of itself.

Appendix B Finite checks

The enumeration finds 26 locked tilings of the 6×4 rectangle, a count obtained here. Their 7 symmetry classes and the 50 locked tilings of the 6×6 rectangle agree with Tucker-Foltz's reported counts [8, Figure 2 and Section 3].

For the finite rigidity checks, only touching pairs need be considered by Lemma 4. The touching pairs occurring in the blocks and seams fall into 162 shapes up to translation and order. Each shape is checked once, then reused by translation invariance. The checking notes describe the cell lists, the complete neighbor lists, the two separately written programs and their checks on incorrect data.

Appendix C Additional illustrations

Figure 4 illustrates a recombination move, Figure 5 the two straight tilings of Lemma 5, and Figure 6 the width-5 constructions of Proposition 9(ii).

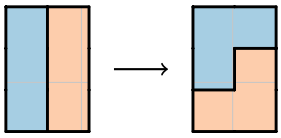


Figure 4: A recombination move: two straight trominoes side by side cover the same six cells as two L-trominoes. A locked tiling has no pair of tiles that allows such a move.

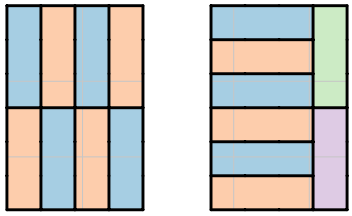


Figure 5: Two tilings of the 6×4 rectangle (Lemma 5). Every rectangle with $3 \mid m$ and $n \geq 3$ has these two, so a tiling all of whose pairs are rigid is locked.

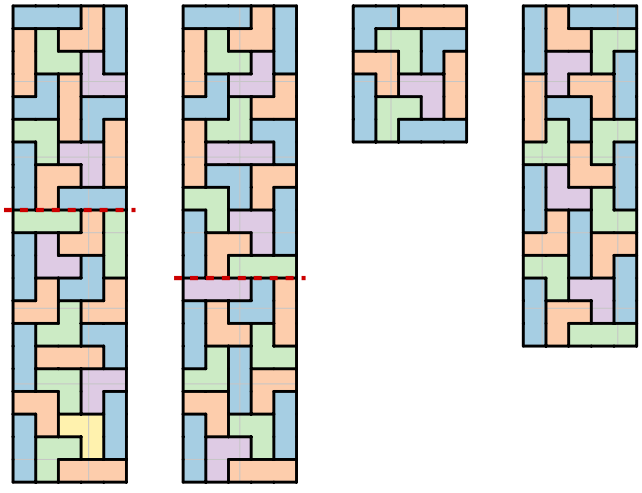


Figure 6: Width 5. Left: the blocks $B_{9,5}$ and $B_{12,5}$ stacked in both orders, seams dashed (each also stacks on itself). Right: the single tilings of the 6×5 and 15×5 rectangles, the two admissible heights that are not sums of nines and twelves.

References

[1] M. Aanjaneya and S. P. Pal, *Faultfree tromino tilings of rectangles*, arXiv:math/0610925, 2006. https://arxiv.org/abs/math/0610925.

[2] H. A. Akitaya, K. Dituro, A. Gonczi, M. Korman, D. L. Souvaine, F. Stock and C. D. Tóth, *Redistricting in triangular and square grids*, in: 2026 Symposium on Simplicity in Algorithms (SOSA), SIAM, 2026, 12–25. doi:10.1137/1.9781611978964.2.

[3] S. Cannon, *Irreducibility of recombination Markov chains in the triangular lattice*, Discrete Applied Mathematics **347** (2024), 75–130. doi:10.1016/j.dam.2023.12.019. Theorem numbering from arXiv:2305.17239v2, December 19, 2023. https://arxiv.org/abs/2305.17239v2.

[4] D. DeFord, M. Duchin and J. Solomon, *Recombination: a family of Markov chains for redistricting*, Harvard Data Science Review **3** (2021), no. 1. doi:10.1162/99608f92.eb30390f.

[5] M. Emre, D. C. Jerison and E. Veomett, *Connectivity of districting metagraphs*, arXiv:2606.10152v2, September 8, 2026. https://arxiv.org/abs/2606.10152v2.

[6] P. Graham, *Locked tromino tilings of rectangles*, version 3, Zenodo, 2026. doi:10.5281/zenodo.23638798.

[7] L. de Moura and S. Ullrich, *The Lean 4 theorem prover and programming language*, in: Automated Deduction – CADE 28, Lecture Notes in Computer Science **12699**, Springer, 2021, 625–635. doi:10.1007/978-3-030-79876-5_37.

[8] J. Tucker-Foltz, *Locked polyomino tilings*, arXiv:2307.15996v4, December 10, 2024. https://arxiv.org/abs/2307.15996v4.