Cyclic sieving for q-Kreweras numbers in the exceptional types

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November 4, 2026

Abstract

Let W be a finite Weyl group with Coxeter number h and simple Lie algebra $\mathfrak{g}$. For each $s \geq 1$ the cyclic group $\mathbb{Z}/sh\mathbb{Z}$ acts on the multichains $w_1 \leq \cdots \leq w_s$ of noncrossing partitions of W. Reiner and Sommers defined, for each nilpotent orbit $\mathcal{O}$ of $\mathfrak{g}$ and each positive integer m, a q-Kreweras number, a polynomial in q, and tabulated it in the exceptional types. Each W-orbit of flats X determines a nilpotent orbit $\mathcal{O}_X$, principal in the Levi subalgebra of X. Their Conjecture 1.4 says that the q-Kreweras number of $\mathcal{O}_X$ with $m = sh + 1$, at a primitive d-th root of unity, counts the multichains fixed by an element of order d whose bottom element has fixed space in the orbit of X. Reiner and Sommers proved it in types A, B, C and D. We prove it in types G_2, F_4, E_6, E_7 and E_8, so it holds for every irreducible Weyl group. At $q = 1$ the statement is a count proved by Douvropoulos and Josuat-Vergès; what is new is every $d > 1$. Krattenthaler and Müller proved the unrefined cyclic sieving, Armstrong's conjecture, in the exceptional types by reducing it to finitely many cases. Using their parametrization of s and of the acting element by three integers, and their description of the fixed points, we show that with two of the integers fixed the fixed-point count is a polynomial of degree at most the rank in the third. A finite-difference argument shows the same for the tabulated q-Kreweras number at the root of unity, once it holds for rank plus one consecutive values of the third integer. Only finitely many choices of the fixed pair need a direct check. The resulting finitely many identities are verified by two independent exact programs, and the reduction is checked in Lean.

This work has not yet been reviewed by experts. The proofs, the Lean formalization, the computer checks and the text were produced by AI models (from OpenAI and Anthropic), working under the author's direction.

Keywords: cyclic sieving, noncrossing partitions, Kreweras numbers, Weyl groups, nilpotent orbits, Fuss–Catalan combinatorics.
MSC 2020: 05E10, 05A15, 20F55, 17B08.

1 Introduction

1.1 Main result

Let Φ be an irreducible crystallographic root system of rank n in a real vector space V, with Weyl group W and Coxeter number h. The *absolute length* $\ell(w)$ of $w \in W$ is the least number of reflections whose product is w; it equals $n - \dim V^w$, where V^w is the space of vectors fixed by w [5, Lemma 2]. The *absolute order* is $u \leq w$ if $\ell(u) + \ell(u^{-1}w) = \ell(w)$. Fix a Coxeter element c, a product of the simple reflections in some order. The noncrossing partitions are $\mathrm{NC}(W) = \{w \in W : w \leq c\}$ [3] (all Coxeter elements are conjugate, and conjugation is an automorphism of the absolute order, so the choice of c does not matter), and $\mathrm{NC}^{(s)}(W)$ is the set of multichains $w_1 \leq w_2 \leq \cdots \leq w_s$ in $\mathrm{NC}(W)$ [2]. Such a multichain is the same as a *delta sequence* $(\delta_0; \delta_1, \ldots, \delta_s)$ with $\delta_0\delta_1 \cdots \delta_s = c$

and $\ell(\delta_0)+\cdots+\ell(\delta_s)=n$, through $\delta_0=w_1$ and $\delta_i=w_i^{-1}w_{i+1}$ ($w_{s+1}=c$). The cyclic group $\mathbb{Z}/sh\mathbb{Z}$ acts on delta sequences through the generator of Armstrong's action, in the form of Bessis and Reiner [4, (6.4)]:

$$\varphi(\delta_0;\delta_1,\dots,\delta_s)=(v;\,c\delta_sc^{-1},\delta_1,\dots,\delta_{s-1}),\qquad v=(c\delta_sc^{-1})\,\delta_0\,(c\delta_sc^{-1})^{-1}. \tag{1}$$

The new bottom element v is conjugate to δ_0, so the W-orbit of the fixed space of the bottom element $w_1=\delta_0$ is constant on $\mathbb{Z}/sh\mathbb{Z}$-orbits.

A *flat* is an intersection of reflecting hyperplanes; each V^w is one, and $[X]=W\cdot X$ denotes the W-orbit of a flat X. Let $\mathfrak{g}$ be the complex simple Lie algebra of Φ with Cartan subalgebra $\mathfrak{h}$. The centralizer of $X\subset\mathfrak{h}$ is a Levi subalgebra $\mathfrak{l}_X$, and $\mathcal{O}_X$ is the nilpotent orbit containing the principal nilpotent elements of $\mathfrak{l}_X$ [13, §1.3]. For a nilpotent orbit $\mathcal{O}$ and a positive integer m, Reiner and Sommers define the q-Kreweras number $\mathrm{Krew}(\Phi,\mathcal{O},m;q)=f_{e,1}(m;q)$, a polynomial in q that they attach to an element $e\in\mathcal{O}$ [13, §1.3]; the conjecture uses $m=sh+1$. They list these polynomials in the exceptional types in the tables of [13, §3.6], and throughout $\mathrm{Krew}(\Phi,\mathcal{O}_X,m;q)$ denotes the formula tabulated there. Write $[x]_q=(1-q^x)/(1-q)$.

Theorem 1.1. *Let W be of type G_2, F_4, E_6, E_7 or E_8. Let $s\geq 1$, let d divide sh, let ω be a primitive d-th root of unity, and let $[X]$ be a W-orbit of flats. Then* $\mathrm{Krew}(\Phi,\mathcal{O}_X,sh+1;q=\omega)$ *is the number of multichains $w_1\leq\cdots\leq w_s$ in* $\mathrm{NC}^{(s)}(W)$ *with $V^{w_1}\in[X]$ that are fixed by an element of order d of $\mathbb{Z}/sh\mathbb{Z}$.*

In the language of Reiner, Stanton and White [15], the theorem says that for each $[X]$ the multichains in $\mathrm{NC}^{(s)}(W)$ with $V^{w_1}\in[X]$, the polynomial $\mathrm{Krew}(\Phi,\mathcal{O}_X,sh+1;q)$ and the cyclic group $\mathbb{Z}/sh\mathbb{Z}$ exhibit the cyclic sieving phenomenon: the polynomial's value at a primitive d-th root of unity counts the points fixed by an element of order d.

Corollary 1.2. *Conjecture 1.4 of Reiner and Sommers [13] holds for every irreducible Weyl group.*

Proof. Reiner and Sommers prove it in types A, B, C, D [13, Theorem 1.7]. In the other types they compute $f_{e,1}$ in the tables of [13, §3.6], which give the function Krew of Theorem 1.1, so Theorem 1.1 proves the conjecture there. □

Example 1.3. Take W of type G_2 ($n=2$, $h=6$) and $s=1$, so $m=7$. Then the generator of $\mathbb{Z}/6\mathbb{Z}$ acts on $\mathrm{NC}(W)$ by conjugation by c, an element of order 6; since $c^3=-1$ is central, c^3 acts trivially. $\mathrm{NC}(W)$ consists of the identity (the flat V), three reflections of each root length and c (the flat $\{0\}$). The four orbits of flats are labeled by the Bala–Carter labels 0, A_1, $\tilde{A}_1$ and G_2 of $\mathcal{O}_X$, and the tables of [13, §3.6] give

$$\tfrac{[m-1]_q[m-5]_q}{[2]_q[6]_q},\quad q^{m-5}\tfrac{[m-1]_q}{[2]_q},\quad q^{m-3}\tfrac{[m-1]_q}{[2]_q},\quad q^{2m-2},$$

which at $m=7$ are 1, $q^2(1+q^2+q^4)$, $q^4(1+q^2+q^4)$ and q^{12}. Their values at a primitive d-th root of unity are:

Label \ d (element)	1 (1)	2 (c^3)	3 (c^2)	6 (c)
0 (the identity)	1	1	1	1
A_1 (long reflections)	3	3	0	0
$\tilde{A}_1$ (short reflections)	3	3	0	0
G_2 (c)	1	1	1	1

Each entry is both the polynomial's value and the number of elements of $\mathrm{NC}(W)$ with that label fixed by conjugation by the element in the column.

The Kreweras numbers refine the Fuss–Catalan number $|\mathrm{NC}^{(s)}(W)|$ according to the W-orbit of the fixed space of the bottom element [13, §1.1]. Reiner and Sommers' q-analogues are attached to nilpotent orbits of $\mathfrak{g}$ [13, §1.3], so the conjecture links the nilpotent orbits of the Lie algebra with the cyclic action on the noncrossing partitions of its Weyl group. With Theorem 1.1 it holds uniformly for every Weyl group.

1.2 Prior work

Reiner and Sommers stated the conjecture in 2016 [13], answering in conjectural form a question that Bessis and Reiner posed for $s = 1$ [4, §6.1, Question 6.3]. Their Conjecture 1.4 reads: "For $m = sh + 1$, and for each W-orbit $[X]$, $\mathrm{Krew}(\Phi, \mathcal{O}_X, m; q = \omega_d)$ counts those $w_1 \leq \cdots \leq w_s \in \mathrm{NC}^{(s)}(W)$ which are fixed under the action an element of order d in the $\mathbb{Z}/sh\mathbb{Z}$-action and have $V^{w_1} \in [X]$." Their Theorem 1.7 proves it in types A, B, C and D. They describe the $q = 1$ count as follows: "In classical (A, B, C, D) and dihedral (I) types, work of Rhoades showed more generally that $\mathrm{Krew}(W, [X], sh + 1)$ counts the elements $w_1 \leq \cdots \leq w_s$ in $\mathrm{NC}^{(s)}(W)$ with V^{w_1} in $[X]$; this remains open in the exceptional types E, F, H" [13, §1.1]. Douvropoulos and Josuat-Vergès proved this count for every irreducible W [6, Theorem 3.10], a theorem they attribute to Rhoades [14] except for E_7 and E_8; for the remaining exceptional groups they write that "this can be done by computer". When $e \in \mathcal{O}_X$, $f_{e,1}(m; 1)$ is this Kreweras number [13, §1.3], so the case $d = 1$ of Theorem 1.1 is their theorem, and what is new here is every $d > 1$. The case $s = 1$ of the count is a theorem of Athanasiadis and Reiner [1, Theorem 6.3], as Reiner and Sommers recall [13, §1.1].

Without the condition on V^{w_1}, the analogous statement for the q-Fuss–Catalan number (a product of ratios of q-integers over the degrees of W) is Armstrong's Conjecture 5.4.7 [2]. Reiner and Sommers record it as their Theorem 1.3, citing Bessis and Reiner [4] [13, §1.2]. Krattenthaler and Müller prove that the generalized noncrossing partitions of the well-generated complex reflection groups of exceptional type "obey two different cyclic sieving phenomena, as conjectured by Armstrong, and by Bessis and Reiner" [8]. To do so they "reduce the conjectures for each of the 26 exceptional types to a finite problem" [8, §1]. Our proof follows that reduction: it uses their formula [8, (5.1)] for powers of φ, the parametrization and the block equations [8, (5.8)] from the proof of their Lemma 30, and their Lemma 31 for large ν.

Three works treat related refinements that Theorem 1.1 does not address. Krattenthaler and Stump pose as an open problem a cyclic sieving phenomenon for positive m-divisible noncrossing partitions with prescribed block structure in exceptional types [9, §7.2, Question 7.5 and Open Problem 7.6]. Pouillart provides "a uniform refinement of these cyclic sieving phenomena using a q-analogue" of the face formula of Douvropoulos and Josuat-Vergès, for faces of the generalized cluster complex under the Fomin–Reading rotation [12]. Kim defines, "for a coincidental Coxeter group, i.e. of type A_{n-1}, BC_n, H_3, or $I_2(m)$", q-Kreweras numbers attached to limit symbols [7]. Our result concerns neither the positive objects, nor faces, nor these other q-analogues.

1.3 Method

The conjecture is a statement for infinitely many s. Following Krattenthaler and Müller, we write $s = m_1\nu$ and the power of φ as $\varphi^{m_1 h_1}$ with $h_1 \mid h$ and $\gcd(h_1, \nu) = 1$. For fixed (h_1, ν) the fixed points with a given $[X]$ are counted by a polynomial in m_1 of degree at most n, built from chains in the centralizer of c^{h_1}, as in the proof of [8, Lemma 30] (Proposition 2.2). The new step for the refinement is the q-Kreweras side: the tabulated q-Kreweras numbers are quotients of products and

sums of q-integers, with no product formula of the kind used for the q-Fuss–Catalan numbers. We show directly from the shape of each formula that if its value at ω is regular and agrees with a polynomial in m_1 of degree at most n for $n+1$ consecutive values of m_1, then it does so for every m_1 (Proposition 2.4). For $\nu > n$ both sides are given in closed form outside finitely many pairs (h_1, ν). The remaining identities, 104 pairs (h_1, ν) with $\nu \le n$ and 21 with $\nu > n$ over the five types, are checked by computer (Section 3), and the reduction is formalized in Lean (Section 4).

2 The reduction to finitely many identities

Write $\mathrm{Fix}_X(s,p)$ for the number of delta sequences of length s fixed by φ^p whose bottom element δ_0 has $V^{\delta_0} \in [X]$, and $R_X(m;q) = \mathrm{Krew}(\Phi, \mathcal{O}_X, m; q)$.

Step 1: the parameters. An element of order d in $\mathbb{Z}/sh\mathbb{Z}$ generates the same subgroup as φ^p with $p = sh/d$, so it has the same fixed points.

Lemma 2.1. *The pairs (s,p) with $s \ge 1$ and $p \mid sh$ correspond one to one to the triples (h_1, ν, m_1) of positive integers with $h_1 \mid h$ and $\gcd(h_1, \nu) = 1$, through $s = m_1\nu$ and $p = m_1 h_1$. Then $d = sh/p = \nu h/h_1$.*

Proof. Given (s,p), put $m_1 = \gcd(p,s)$, $\nu = s/m_1$ and $h_1 = p/m_1$. Then $\gcd(h_1,\nu) = 1$, and $p \mid sh$ gives $h_1 \mid \nu h$, hence $h_1 \mid h$. Conversely $s = m_1\nu$ and $p = m_1 h_1$ satisfy $p \mid sh$ and $\gcd(p,s) = m_1$. □

Step 2: the fixed points. Fix $h_1 \mid h$ and $\nu \ge 1$ with $\gcd(h_1,\nu) = 1$, and write $h_1 = t\nu + r$ with $0 \le r < \nu$, as in the proof of [8, Lemma 30]. Since $\gcd(r,\nu) = 1$, the map sending k to $k+r$, or to $k+r-\nu$ when $k+r > \nu$, is one cycle on $\{1,\dots,\nu\}$. Starting from $e_1 = 0$, follow this cycle for $\nu - 1$ steps, setting $e_{k+r} = e_k + t$ for a step with $k + r \le \nu$ and $e_{k+r-\nu} = e_k + t + 1$ for a step with $k + r > \nu$. This defines $e_1, \dots, e_\nu$. Going once round the cycle would add $\nu t + r = h_1$, so the step that closes it defines nothing; in the proof below it becomes the condition of commuting with c^{h_1}. Equivalently, e_k is the unique integer in $[0, h_1)$ with $h_1 \mid \nu e_k + (k-1)$, and for elements commuting with c^{h_1} any exponents with this divisibility give the same conjugates below (this is the form used in Appendix B). Let C be the centralizer of c^{h_1} in W. For $u \in \mathrm{NC}(W) \cap C$ put $u_k = c^{e_k} u c^{-e_k}$ and $\delta_0(u) = c\,(u_1 u_2 \cdots u_\nu)^{-1}$, and let

$$U_X = \{u \in \mathrm{NC}(W) \cap C : \ \ell(\delta_0(u)) + \nu\,\ell(u) = n,\ V^{\delta_0(u)} \in [X]\}.$$

For $u \in \mathrm{NC}(W)$ let $N_k(u)$ be the number of chains $1 = z_0 < z_1 < \cdots < z_k = u$ in the absolute order with every $z_i \in C$.

Proposition 2.2. *For every $m_1 \ge 1$,*

$$\mathrm{Fix}_X(m_1\nu,\, m_1 h_1) = L_X(m_1) := \sum_{u \in U_X} \sum_{k=0}^{n} \binom{m_1}{k} N_k(u).$$

In particular L_X is a polynomial in m_1 of degree at most n.

Proof. Put $s = m_1\nu$ and $p = m_1 h_1 = ts + m_1 r$. On the last s entries, φ acts by $\rho(\delta_1,\dots,\delta_s) = (c\delta_s c^{-1}, \delta_1, \dots, \delta_{s-1})$, and $\delta_0 = c(\delta_1 \cdots \delta_s)^{-1}$ is determined by them, so the fixed points of φ^p are those of ρ^p. Since ρ^s is conjugation by c in each entry, $\rho^p(\delta)_i = c^{t+1}\delta_{i-m_1 r+s}c^{-t-1}$ for $i \le m_1 r$ and $\rho^p(\delta)_i = c^t \delta_{i-m_1 r} c^{-t}$ for $i > m_1 r$ [8, (5.1), (5.8)]. Cut $(\delta_1,\dots,\delta_s)$ into ν blocks $B_1, \dots, B_\nu$

of m_1 consecutive entries. As the shift $m_1 r$ is a multiple of m_1, the equation $\rho^p\delta = \delta$ says that $B_{k+r} = c^t B_k c^{-t}$ when $k + r \le \nu$ and $B_{k+r-\nu} = c^{t+1} B_k c^{-t-1}$ otherwise, entry by entry. Following the cycle from B_1, as in the proof of [8, Lemma 30], these equations hold exactly when every entry of B_1 commutes with c^{h_1} and $B_k = c^{e_k} B_1 c^{-e_k}$ for all k.

Let u be the product of the entries of B_1. The product of the entries of B_k is then u_k, so $\delta_0 = \delta_0(u)$, and by subadditivity and conjugation invariance of ℓ,

$$n = \ell(\delta_0) + \nu \sum_{x \in B_1} \ell(x) \;\ge\; \ell(\delta_0) + \nu\,\ell(u) = \ell(\delta_0) + \sum_k \ell(u_k) \;\ge\; \ell(c) = n.$$

Both inequalities are equalities: B_1 is a length-additive factorization of u into m_1 elements of C, and $\ell(\delta_0(u)) + \nu\ell(u) = n$. The second condition makes $c = \delta_0 u_1 \cdots u_\nu$ length-additive, so $u = u_1 \le c$ and $u \in \mathrm{NC}(W) \cap C$. Conversely, each $u \in \mathrm{NC}(W) \cap C$ with $\ell(\delta_0(u)) + \nu\ell(u) = n$ and each length-additive factorization of u into m_1 elements of C gives a fixed point. Length-additive factorizations $u = x_1 \cdots x_{m_1}$ correspond to multichains $1 = y_0 \le y_1 \le \cdots \le y_{m_1} = u$ through $y_j = x_1 \cdots x_j$, and all x_j lie in C exactly when all y_j do. A multichain with exactly k strict steps is a strict chain of length k together with a choice of the k strict positions among m_1, which gives $\binom{m_1}{k} N_k(u)$. □

Lemma 2.3 ([8, Lemma 31])**.** *If $\nu > n$, then $L_X(m_1) = 1$ when $X = \{0\}$ and $L_X(m_1) = 0$ otherwise.*

Proof. If $u \ne 1$ then $\ell(\delta_0(u)) + \nu\ell(u) \ge \nu > n$, so $U_X \subseteq \{1\}$. For $u = 1$, $\delta_0 = c$, whose fixed space is $\{0\}$, and $N_k(1)$ is 1 for $k = 0$ and 0 for $k > 0$. □

Step 3: the right side. Every formula $R_X(m; q)$ in the tables of [13, §3.6] that we use is a product of factors of four kinds: $q^{\alpha m + \beta}$ with integers α, β; numerator brackets $[m - a]_q$; sums $\sum_j \pm q^{\gamma_j}[m - a_j]_q$ of such brackets; and denominator brackets $1/[b]_q$ with $b \ge 1$. In each tabulated formula (a row of the tables) there are at most n factors of the second and third kinds together, and at most n of the fourth kind; the programs check this for every row.

Proposition 2.4. *Let $R(m; q)$ be a product of factors of the four kinds, with at most n of the second and third kinds together and at most n of the fourth kind. Let $E \ge 1$, and let ω be a root of unity with $\omega^E = 1$. Suppose that for $m_1 = 1, \ldots, n + 1$ the rational function $R(Em_1 + 1; q)$ is regular at $q = \omega$ with value $P(m_1)$, where P is a polynomial of degree at most n. Then the same holds for every $m_1 \ge 1$.*

Proof. Put $Z = q^{Em_1}$, so that $q^m = qZ$ for $m = Em_1 + 1$. Then $q^{\alpha m + \beta} = q^{\alpha+\beta} Z^\alpha$ and $[m - a]_q = (1 - q^{1-a}Z)/(1 - q)$. Multiplying out, $R(Em_1 + 1; q) = \sum_i \psi_i(q) Z^{\alpha_i}$ with finitely many integers α_i and rational functions ψ_i that do not depend on m_1. Each ψ_i has a pole of order at most n at ω: if $\omega = 1$ the poles come only from the factors $1/(1 - q)$, one for each factor of the second or third kind, and if $\omega \ne 1$ only from the denominator brackets, each of which has at most a simple zero at ω. Let Δ be the forward difference in m_1. Since $\Delta Z^\alpha = Z^\alpha(q^{E\alpha} - 1)$,

$$\Delta^{n+1} R(Em_1 + 1; q) = \sum_i \psi_i(q)\, Z^{\alpha_i}\, (q^{E\alpha_i} - 1)^{n+1}.$$

As $\omega^E = 1$, each factor $(q^{E\alpha_i} - 1)^{n+1}$ vanishes to order at least $n + 1$ at ω, so every term is regular at ω with value 0. Now $R(E(m_1 + n + 1) + 1; q)$ equals $\Delta^{n+1} R(Em_1 + 1; q)$ minus a fixed integer combination of $R(E(m_1 + j) + 1; q)$ for $0 \le j \le n$, and P satisfies the same relation with $\Delta^{n+1} P = 0$. Induction on m_1 completes the proof. □

Step 4: proof of Theorem 1.1. Let s, d and $[X]$ be given, and let (h_1, ν, m_1) correspond to $(s, sh/d)$ by Lemma 2.1. Put $E = \nu h$, so $sh+1 = Em_1+1$ and $\omega^E = 1$ because $d = \nu h/h_1$ divides E. By Proposition 2.2 the left side is $L_X(m_1)$, a polynomial of degree at most n.

If $\nu \le n$, there are finitely many pairs (h_1, ν), and for each of them and each $[X]$ the computation of Section 3 verifies that $R_X(Em_1+1; q)$ is regular at ω with value $L_X(m_1)$ for $m_1 = 1, \dots, n+2$. Proposition 2.4 gives every m_1.

If $\nu > n$, then by Lemma 2.3 L_X is 1 for $X = \{0\}$ and 0 otherwise. Let $b_{\max}$ be the largest b of a denominator bracket in the type's table (6, 12, 12, 18 and 30 for G_2, F_4, E_6, E_7, E_8). If $d > b_{\max}$, no denominator bracket vanishes at ω. Every row except that of $X = \{0\}$ has a factor $[m-1]_q$, which vanishes at $\omega \neq 1$ because $\omega^{m-1} = \omega^{Em_1} = 1$, and the row of $X = \{0\}$ is $q^{n(m-1)}$, which is 1 at ω; the programs check both facts. If $d \le b_{\max}$, only finitely many pairs (h_1, ν) occur, and the computation verifies that $R_X(Em_1+1; \omega)$ is 1 for $X = \{0\}$ and 0 otherwise, for $m_1 = 1, \dots, n+2$; again Proposition 2.4 gives every m_1. This covers every s, every $d \mid sh$ and every $[X]$. □

3 Computational verification

For each type the programs build Φ, the reflections as integer matrices and $c = s_1 \cdots s_n$ in Bourbaki's numbering, with $\ell(w) = \operatorname{rank}(w-1)$ [5, Lemma 2]. They compute $\mathrm{NC}(W)$ and check that its size is the Catalan number of W (8, 105, 833, 4160, 25080), and the absolute order on it, whose number of comparable pairs $u \le w$ is checked against the Fuss–Catalan number $\mathrm{Cat}^{(2)}(W)$.

The programs then match each W-orbit of flats with its row in the tables of [13, §3.6] and check that the match is a bijection onto the 4, 12, 17, 32 and 41 rows of orbits principal in a Levi subalgebra (Bala–Carter labels without parentheses), taken with the trivial representation of the component group. Two invariants decide the match. The type of the parabolic subgroup fixing the flat must be the Levi type of the label. The number of hyperplanes of the arrangement restricted to the flat must be the sum of its Orlik–Solomon exponents; these are the representation exponents listed in the second column of all the rows of the orbit [13, §2 after (2.2), and §3.6], and for ten orbits one of them is listed in a row with a nontrivial representation. The second invariant separates three pairs of orbits in E_7 with the same Levi type. These 106 rows were transcribed by a program from the arXiv source of [13] and compared with the journal pages.

For each $h_1 \mid h$ and $1 \le \nu \le n$ with $\gcd(h_1, \nu) = 1$, and for the pairs with $\nu > n$ and $d \le b_{\max}$ (Appendix A), a certificate lists the coefficients of L_X in the basis $\binom{m_1}{k}$ for every $[X]$. Two checkers, written separately and sharing no code, recompute everything from the root system and compare with the certificate:

- Checker A builds $\mathrm{NC}(W)$ bottom-up with exact ranks, finds a standard parabolic subgroup conjugate to the one fixing each flat, and evaluates each tabulated formula as an exact polynomial in $\mathbb{Z}[q]$ (the division in each formula is checked exact), reduced modulo the cyclotomic polynomial of order d; the residue must be the integer $L_X(m_1)$.

- Checker B lets W act by permutations of the roots, builds $\mathrm{NC}(W)$ top-down from c, identifies orbits of flats by the type of the roots vanishing on V^w and the number of restricted hyperplanes, derives the exponents e_k by iterating ρ symbolically, and evaluates each tabulated formula from the leading term of each factor's expansion around $q = \omega$, in $\mathbb{Q}(e^{2\pi i/d})$.

Both check every identity for $m_1 = 1, \dots, n+2$ and the row shapes used in Step 3. Checker B also recounts fixed points by brute force, by the literal action (1) and by ρ, for the small s given in Appendix A. The identities with $d = 1$ are the counts of [6, Theorem 3.10], and they agree for

all s. Deliberately corrupted certificates and tables are rejected by both checkers. The README gives the commands.

4 Formal verification

Steps 1–4 are proved in Lean 4 [10] with Mathlib [11]. The theorem `RS16.Main.conjecture_1_4` has the following form.

Data: a finite group G; a function $\ell : G \to \mathbb{N}$; an element $c \in G$ and integers h and n; a map cls from G to a set of labels; for each label X a formula $\mathrm{rows}(X)$ of the shape of Step 3; and block exponents e.

Hypotheses (each matched to its check in Appendix B):

- ℓ is subadditive, invariant under conjugation and zero only at 1;
- $c^h = 1$ and $\ell(c) = n > 0$;
- the exponent condition of Step 2;
- the row shapes and the bound $b_{\max}$;
- the row of the label of c is $q^{n(m-1)}$, and every other row has a factor $[m-1]_q$;
- the identities at $m_1 = 1, \dots, n+1$ for $\nu \le n$, and those for $\nu > n$ and $d \le b_{\max}$.

Conclusion: for every $s \ge 1$, every $k \in \mathbb{Z}/sh\mathbb{Z}$ of additive order d, every label X and every primitive d-th root of unity ω in a field of characteristic 0, the formula $\mathrm{rows}(X)$ at $m = sh + 1$ is regular at ω, and its value there is the number of delta sequences fixed by φ^k whose bottom element has label X.

Instantiated with $G = W$, $\ell(w) = \mathrm{rank}(w-1)$ and $\mathrm{cls}(w)$ the label of $W \cdot V^w$, this is Theorem 1.1. Lean also proves that $\mathrm{rank}(w-1)$ on a group of invertible matrices satisfies the three conditions on ℓ. The eleven theorems of the formal statement depend only on the axioms `propext`, `Classical.choice` and `Quot.sound`.

Outside Lean remain: the finite computations themselves; that the checkers' rational-function arithmetic computes the formal evaluation of a row, and that their $\mathrm{NC}(W)$, order and length are the objects defined above, and that their fixed-point polynomial is the L_X of Step 2 (with U_X, the chain counts and the block exponents as defined there), which a reader confirms from the code; the identification of reflection length with $\mathrm{rank}(w-1)$ [5, Lemma 2]; the formulas of [13, §3.6] as stated there, and that each row the checkers read is the formula of the table; and the labeling of orbits of flats. Appendix B matches each hypothesis to its check.

Appendix A The cases checked

For each type the table gives the rank n, the Coxeter number h, the size of $\mathrm{NC}(W)$, the number of orbits of flats, the number of pairs (h_1, ν) with $h_1 \mid h$, $1 \le \nu \le n$ and $\gcd(h_1, \nu) = 1$, the number of pairs with $\nu > n$ and $\nu h/h_1 \le b_{\max}$, $b_{\max}$, and the largest s for which checker B recounts fixed points by brute force, with the literal action (1) and with ρ.

Type	n	h	$\lvert\mathrm{NC}(W)\rvert$	orbits	pairs, $\nu \le n$	pairs, $\nu > n$	$b_{\max}$	s, (1)	s, ρ
G_2	2	6	8	4	6	1	6	4	4
F_4	4	12	105	12	13	4	12	3	4
E_6	6	12	833	17	20	2	12	2	3
E_7	7	18	4160	32	27	4	18	1	2
E_8	8	30	25080	41	38	10	30	1	2

Appendix B The formal statement and the checks

The hypotheses of `RS16.Main.conjecture_1_4` and what establishes each:

- subadditivity, conjugation invariance and $\ell(w) = 0 \iff w = 1$ for $\ell(w) = \mathrm{rank}(w-1)$: proved in Lean (`reflection_length`);
- $c^h = 1$ and $\ell(c) = n$: a separate program computes the order of c and the rank of $c-1$ in each type;
- the exponent condition $h_1 \mid \nu e_{j+1} + j$ for $0 \le j < \nu$ (the form of Step 2) for the block exponents: proved in Lean for the recursion that checker A uses (`checker_exponents`), and such exponents exist for every coprime pair (`exists_exponents`);
- at most n numerator and n denominator brackets per row, the row $q^{n(m-1)}$, the factor $[m-1]_q$ and the bound $b_{\max}$: checked on every row by both checkers;
- the identities for $\nu \le n$ and for $\nu > n$ with $d \le b_{\max}$: checked by both checkers; a constant residue modulo the cyclotomic polynomial gives the value at every primitive d-th root (`checker_value`).

Lean proves Step 2 (`step2_fixed_points`, `step2_large_m2`), Step 3 (`step3_rows`), the vanishing of rows with a factor $[m-1]_q$ (`rows_with_m1_factor`) and Step 1 inside `conjecture_1_4`. Step 3 is proved by the finite differences above, so it uses neither the positivity of $f_{e,1}$ [13, Theorem 1.6(iv)] nor the check at $m_1 = n+2$. In the Lean statement a multichain is represented by its delta sequence $(\delta_0; \delta_1, \ldots, \delta_s)$, with $\delta_0 = w_1$ as in Section 1.1.

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