A subdivided 6-clique breaks a conjectured sum bound for independent monophonic-position colorings

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Abstract

A set of vertices of a graph is in monophonic position if no induced path contains three of its vertices. Ullas Chandran S. V., Gabriele Di Stefano, S. Haritha, Elias John Thomas and James Tuite introduced colorings whose color classes are such sets, and the variant whose classes are also independent. They showed that the monophonic-position chromatic numbers of an n-vertex graph and of its complement sum to at most $n+1$, and conjectured the same Nordhaus–Gaddum bound for the independent variant; they report a computer search by Grahame Erskine confirming their conjecture for all graphs with at most ten vertices. We show that the independent monophonic sum bound fails. Subdividing every edge of the complete graph K_6 once gives a graph H on 21 vertices that needs at least eight independent monophonic-position color classes, while its complement needs at least fifteen, so the sum is at least 23. The proof classifies the independent monophonic-position sets of H and shows by counting that seven color classes cannot cover H. The lower bounds are also confirmed by SAT-solver proofs checked with a formally verified checker.

This work has not yet been reviewed by experts. The proofs, the code and the text were produced by AI models (from OpenAI and Anthropic), working under the author's direction.

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1 Introduction

All graphs are finite, simple and undirected. An induced path has distinct vertices, with adjacency precisely between consecutive vertices. A monophonic-position set meets every induced path in at most two vertices. Following Chandran, Di Stefano, Haritha, Thomas and Tuite [1, Definition 2.3], write $\chi_{\mathrm{mp_i}}(G)$ for the minimum number of independent monophonic-position sets partitioning $V(G)$ (color classes may be empty), the independent monophonic-position chromatic number of G. Dropping independence gives the monophonic-position chromatic number $\chi_{\mathrm{mp}}(G)$, and replacing induced paths by shortest paths gives the general-position analogues. Both properties pass to subsets: a subset of an independent monophonic-position set is again one.

Chandran et al. proposed Nordhaus–Gaddum relations for these colorings, after the classical bound $\chi(G) + \chi(\overline{G}) \leq n + 1$ for the chromatic number [2]. From their Lemma 4.1, the remark before their Conjecture 9.3 derives, for every graph G of order n,

$$\chi_{\mathrm{mp}}(G) + \chi_{\mathrm{mp}}(\overline{G}) \leq n + 1,$$

and the same for general position; it adds that "up to order nine the only example of equality is C_5 for χ_{mp}". Conjecture 9.3 asserts that for general position and for monophonic position alike, the independent parameters satisfy the same sum bound and a product bound. For monophonic position its sum part reads

$$\chi_{\mathrm{mp_i}}(G) + \chi_{\mathrm{mp_i}}(\overline{G}) \le n + 1. \tag{1}$$

The paragraph after the conjecture states: "It is true for all graphs with up to ten vertices, as shown by a computer search by Erskine", citing a 2021 private communication from Grahame Erskine. The statements cited here are numbered alike in arXiv version 2 and the published version of [1].

Let H be the graph obtained from K_6 by subdividing every edge once. Its original vertices form $A = \{0, 1, \ldots, 5\}$ and its subdivision vertices form $B = \{b_{ij} : 0 \le i < j \le 5\}$, where we write $b_{ji} = b_{ij}$; each b_{ij} is adjacent exactly to i and j. Thus H has 21 vertices and 30 edges (Figure 1). Both H and $\overline{H}$ are connected, so the convention of [1] for disconnected graphs plays no role.

Theorem 1. *For the graph H just defined,*

$$\chi_{\mathrm{mp_i}}(H) \ge 8, \qquad \chi_{\mathrm{mp_i}}(\overline{H}) \ge 15.$$

Consequently $\chi_{\mathrm{mp_i}}(H) + \chi_{\mathrm{mp_i}}(\overline{H}) \ge 23 > |V(H)| + 1$, so H violates (1).

Theorem 1 disproves the monophonic sum part (1) of Conjecture 9.3; the general-position sum part and both product parts remain open. Thus, for monophonic position, the sum bound proved without independence does not extend to the independent parameter. The counterexample needs only these lower bounds; the remark in Section 2 gives colorings attaining them. We do not know whether a smaller counterexample exists. Monophonic position sets in Cartesian and lexicographic products are studied in [3]; we do not use those results. Appendix A lists further related work.

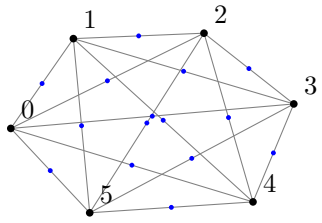


Figure 1: The graph H. Original vertices are black; subdivision vertices are blue, each joined to the two original vertices of the edge of K_6 it subdivides. Crossings without a dot are not vertices.

2 Proof of Theorem 1

The main step is a classification: every independent monophonic-position set of H with at least three vertices lies among the five subdivision vertices adjacent to a single original vertex. The lower bound for H then follows by counting how many original and subdivision vertices a collection of such sets can cover. The lower bound for $\overline{H}$ is immediate, because B is a clique there.

For $i \in A$, let $C_i = N_H(i) = \{b_{ij} : j \in A \setminus \{i\}\}$ be the star set at i, the five subdivision vertices adjacent to i. We call $\mathcal{C}$ the catalog: it consists of the six star sets and all independent pairs of H not contained in a star set. There are 120 such pairs: H has $\binom{21}{2} - 30 = 180$ independent pairs, and the star sets contain $6\binom{5}{2} = 60$ distinct pairs.

Lemma 2 (Classification of color classes)**.** *Each member of $\mathcal{C}$ is an independent monophonic-position set of H. Every independent monophonic-position set of H is contained in a member of $\mathcal{C}$.*

Proof. A set of at most two vertices meets every path in at most two vertices, so every independent pair of H is an independent monophonic-position set. Each C_i is independent. If an induced path omits i, any vertex of C_i on it must be an endpoint, since an interior b_{ij} needs both neighbors i, j. If the path contains i, all its vertices from C_i must be consecutive to i, to avoid chords. In either case it contains at most two vertices of C_i.

We show that every independent triple not contained in a star set lies on an induced path. A simple path in K_6, with its edges subdivided, is induced in H. One may also append a subdivision vertex at either end when its other neighbor is absent from the path. Up to renaming, an independent triple that meets A has one of the four forms below, in which distinct letters denote distinct original vertices: independence forces each subdivision vertex of the triple to avoid the triple's original vertices, and two subdivision vertices share one original vertex or none. Each row gives an induced path containing the triple:

Triple	Induced path
$\{a, a', c\}$	$a, b_{ac}, c, b_{ca'}, a'$
$\{a, a', b_{cd}\}$	$a, b_{ac}, c, b_{cd}, d, b_{da'}, a'$
$\{a, b_{cd}, b_{df}\}$	$a, b_{ac}, c, b_{cd}, d, b_{df}, f$
$\{a, b_{cd}, b_{ef}\}$	$a, b_{ac}, c, b_{cd}, d, b_{de}, e, b_{ef}$

A triple in B corresponds to three edges of K_6. Unless they share a common endpoint, their graph is a triangle, a three-edge path, a two-edge path and a disjoint edge, or three disjoint edges. The required paths, in the same order, are

Edges of K_6	Induced path in H
xy, xz, yz	$b_{xy}, x, b_{xz}, z, b_{yz}$
xy, yz, zt	$x, b_{xy}, y, b_{yz}, z, b_{zt}, t$
xy, yz, uv	$x, b_{xy}, y, b_{yz}, z, b_{zu}, u, b_{uv}, v$
xy, zu, vw	$x, b_{xy}, y, b_{yz}, z, b_{zu}, u, b_{uv}, v, b_{vw}, w$

Thus every triple in an independent monophonic-position set S lies in a star set. If $|S| \geq 3$, fix two elements of one such triple and its star set C_i. For each remaining element of S, the triple formed with that pair lies in a star set. Since $|C_i \cap C_j| = 1$ for $i \neq j$, that star set must again be C_i. Hence $S \subseteq C_i$. Sets of size at most two are contained in members of $\mathcal{C}$ as well. □

Proof of Theorem 1. By Lemma 2, extending every color class of a coloring of H to a member of $\mathcal{C}$ containing it gives a cover of $V(H)$ by no more members of $\mathcal{C}$ than colors. Remove repeated members. Suppose the cover uses c star sets and p_{AA}, p_{AB}, p_{BB} pairs with, respectively, both vertices in A, one vertex in each of A and B, and both vertices in B; put $p = p_{AA} + p_{AB} + p_{BB}$. The star sets lie in B, so only pairs cover A: $2p_{AA} + p_{AB} \geq 6$. The subdivision vertices whose two original endpoints are outside the c star centers number $\binom{6-c}{2}$. They too must be covered by pairs, giving $p_{AB} + 2p_{BB} \geq \binom{6-c}{2}$. Adding yields

$$c + p \;\geq\; c + 3 + \frac{1}{2}\binom{6-c}{2} = \frac{15}{2} + \frac{(c-3)(c-4)}{4} \;\geq\; \frac{15}{2},$$

since c is an integer between zero and six. The number of classes is an integer, so $\chi_{\mathrm{mp_i}}(H) \geq 8$.

In $\overline{H}$, the fifteen vertices of B form a clique, so an independent set of $\overline{H}$ contains at most one of them. Hence $\chi_{\mathrm{mp_i}}(\overline{H}) \geq 15$. □

Remark. Both bounds are attained. Partition B into the five nonempty sets $D_i = C_i \setminus \bigcup_{j<i} C_j$ for $0 \le i \le 4$, and partition A into $\{0,1\},\{2,3\},\{4,5\}$. Each D_i is a subset of a star set, so by Lemma 2 and heredity these eight classes are independent monophonic-position sets. Every b_{ij} with $i < j$ belongs to D_i, so they cover all vertices, and $\chi_{\mathrm{mp_i}}(H) = 8$. In $\overline{H}$, take the six classes

$$\{0, b_{01}\},\ \{1, b_{12}\},\ \{2, b_{23}\},\ \{3, b_{34}\},\ \{4, b_{45}\},\ \{5, b_{05}\},$$

and the nine remaining vertices of B as singletons. Each pair is an edge of H, hence independent in $\overline{H}$, and any set of size at most two is a monophonic-position set. These fifteen classes give $\chi_{\mathrm{mp_i}}(\overline{H}) = 15$.

3 Verification and reproducibility

The proof above is complete by hand. As an independent check, the completeness of $\mathcal{C}$ and the absence of a cover of H by seven members of $\mathcal{C}$ are encoded as Boolean satisfiability problems, the cardinality bound by Sinz's sequential counter [19]; the same is done for $\overline{H}$, with its catalog of the 30 edges of H and covers by fourteen members. The SAT solver CaDiCaL [16] produced proofs of unsatisfiability. Wetzler, Heule and Hunt's drat-trim [17] checks them and converts them to LRAT proofs, which Tan, Heule and Myreen's cake_lpr [18], a checker verified in the CakeML system, checks independently. The verified checker checks propositional proofs, not their graph-theoretic interpretation; that step relies on the Python code that builds the formulas from H. The code, the exact encodings, the certificates and the instructions are in the folder `checks/` of the result's repository.

Appendix A Related work

The prior-work search was bounded, and this list is selected; the checking notes in the result's repository (`checks/README.md`) record the versions read and not read and why each source is excluded. In the versions read, none of the following works treats the sum (1).

1. Thomas, Chandran, Tuite and Di Stefano [4]: Section 5 defines the "independent monophonic position" number, the largest size of one such set.
2. Thomas and Chandran [5]: "An independent set S of vertices in a graph G is an independent position set if no three vertices of S lie on a common geodesic."
3. Sadiquali and Sudhahar [6], a different parameter under a similar name: "the minimum size of an independent monophonic set".
4. Chandran, Klavžar, Neethu and Tuite [3]: "the monophonic position number is studied on Cartesian and lexicographic products of graphs".
5. Tuite, Thomas and Chandran [7]: "we investigate some extremal problems for these parameters".
6. Di Stefano, Klavžar, Krishnakumar, Tuite and Yero [8]: the "smallest maximal general position set".
7. Klavžar, Tian and Tuite [9]: "a general position set is built through adversarial play".
8. Thomas, Chandran, Tuite and Di Stefano [10]: "we investigate the general position numbers of the Mycielskian of graphs".

9. Iršič, Klavžar, Rus and Tuite [11]: coefficients giving "the number of distinct general position sets of G with cardinality" i.
10. Kruft Welton, Khudairi and Tuite [12]: "the lower general position problem asks for a smallest maximal general position set".
11. Araujo, Dourado, Protti and Sampaio [13], on the iteration time and the general position number in graph convexities.
12. Hamed-Labbafian, Henning and Tavakoli [14], on edge general position.
13. Yan [15]: "No circulant graph of order 15 has both diameter 2 and monophonic position number 2".

References

[1] Ullas Chandran S. V., Gabriele Di Stefano, S. Haritha, Elias John Thomas and James Tuite, *Colouring a graph with position sets*, Ars Mathematica Contemporanea 26 (2026), #P4.06. doi:10.26493/1855-3974.3454.a3c. arXiv:2408.13494v2.

[2] E. A. Nordhaus and J. W. Gaddum, *On complementary graphs*, American Mathematical Monthly 63 (1956), 175–177. doi:10.2307/2306658.

[3] Ullas Chandran S. V., Sandi Klavžar, P. K. Neethu and James Tuite, *Monophonic position sets of Cartesian and lexicographic products of graphs*, Computational and Applied Mathematics 46 (2027), article 44; published online September 11, 2026. doi:10.1007/s40314-026-03901-3. arXiv:2412.09837v3.

[4] Elias John Thomas, Ullas Chandran S. V., James Tuite and Gabriele Di Stefano, *On monophonic position sets in graphs*, Discrete Applied Mathematics 354 (2024), 72–82. doi:10.1016/j.dam.2023.02.021. arXiv:2012.10330v4.

[5] Elias John Thomas and Ullas Chandran S. V., *On independent position sets in graphs*, Proyecciones 40(2) (2021), 385–398. doi:10.22199/issn.0717-6279-2021-02-0023.

[6] A. Sadiquali and P. Arul Paul Sudhahar, *Independent Monophonic Sets in Graphs*, Mapana Journal of Sciences 18(3) (2019), 33–44. doi:10.12723/mjs.50.3.

[7] James Tuite, Elias John Thomas and Ullas Chandran S. V., *On some extremal position problems for graphs*, Ars Mathematica Contemporanea 25 (2025), #P1.09. doi:10.26493/1855-3974.3094.bc6. arXiv:2106.06827.

[8] Gabriele Di Stefano, Sandi Klavžar, Aditi Krishnakumar, James Tuite and Ismael G. Yero, *Lower general position sets in graphs*, Discussiones Mathematicae Graph Theory 45(2) (2025), from page 509. doi:10.7151/dmgt.2542.

[9] Sandi Klavžar, Jing Tian and James Tuite, *Builder-Blocker general position games*, Communications in Combinatorics and Optimization 11(2) (2026), 465–486. doi:10.22049/cco.2024.29122.1854. arXiv:2310.19949v2.

[10] Elias John Thomas, Ullas Chandran S. V., James Tuite and Gabriele Di Stefano, *On the general position number of Mycielskian graphs*, Discrete Applied Mathematics 353 (2024), 29–43. doi:10.1016/j.dam.2024.03.015. arXiv:2203.08170v2.

[11] Vesna Iršič, Sandi Klavžar, Gregor Rus and James Tuite, *General position polynomials*, Results in Mathematics 79 (2024), article 110. doi:10.1007/s00025-024-02133-3.

[12] Eartha Kruft Welton, Sharif Khudairi and James Tuite, *Lower general position in Cartesian products*, Communications in Combinatorics and Optimization 10(1) (2025), 111–126. doi:10.22049/cco.2024.29171.1879. arXiv:2404.19451v1.

[13] Julio Araujo, Mitre C. Dourado, Fábio Protti and Rudini Sampaio, *The iteration time and the general position number in graph convexities*, Applied Mathematics and Computation 487 (2025), 129084. doi:10.1016/j.amc.2024.129084. arXiv:2305.00467v5.

[14] Zahra Hamed-Labbafian, Michael A. Henning and Mostafa Tavakoli, *Edge general position in graphs: Graph products, integer linear programming and some applications*, Discrete Applied Mathematics 386 (2026), 1–8. doi:10.1016/j.dam.2026.01.008.

[15] Zeyu Yan, *Monophonic Position in Circulant Graphs: A Counterexample, an Induced-Path Criterion, and an Infinite Family*, August 18, 2026. doi:10.5281/zenodo.21995203.

[16] Armin Biere, Tobias Faller, Katalin Fazekas, Mathias Fleury, Nils Froleyks and Florian Pollitt, *CaDiCaL 2.0*, CAV 2024, Lecture Notes in Computer Science 14681, 133–152. Author manuscript.

[17] Nathan Wetzler, Marijn J. H. Heule and Warren A. Hunt, Jr., *DRAT-trim: Efficient Checking and Trimming Using Expressive Clausal Proofs*, SAT 2014, Lecture Notes in Computer Science 8561, 422–429. Author manuscript.

[18] Yong Kiam Tan, Marijn J. H. Heule and Magnus O. Myreen, *cake_lpr: Verified Propagation Redundancy Checking in CakeML*, TACAS 2021, Lecture Notes in Computer Science 12652, 223–241. doi:10.1007/978-3-030-72013-1_12.

[19] Carsten Sinz, *Towards an Optimal CNF Encoding of Boolean Cardinality Constraints*, CP 2005, Lecture Notes in Computer Science 3709, 827–831. doi:10.1007/11564751_73.