The Heawood family is not the complete list of excluded minors for 4-flat graphs

P. Graham
https://orcid.org/0009-0007-8493-1505

November 11, 2026

Abstract

A graph is 4-flat if the complex obtained by filling every simple cycle with a separate disk embeds piecewise linearly in four-dimensional space. The Heawood family consists of 78 graphs obtained from K_7 and $K_{3,3,1,1}$ by triangle–star and star–triangle transformations. Van der Holst proved that these graphs are not 4-flat; Georgakopoulos and Winter completed the proof that all 78 are excluded minors and conjectured that there are no other excluded minors for 4-flatness. We show that the known eleven-vertex, twenty-two-edge graph H is not 4-flat and has no Heawood-family minor, resolving the completeness question negatively. Mattman and Pierce found by computer search that every proper minor of H is 2-apex; this implies that H itself is a further excluded minor. Mattman and Pierce found H, subsequently studied by Kim and Mattman. An earlier paper by the author proved that its one-edge extension G_1 has Colin de Verdière invariant 6 and no Heawood-family minor. This made 4-flatness completeness incompatible with van der Holst's proposed equivalence between 4-flatness and invariant at most 5. The new step is an explicit mod-two deleted-product cycle with invariant one. Van der Holst and Pendavingh's obstruction theorem therefore excludes an even map, and hence an embedding, of a cycle subcomplex of H into four-dimensional space. Two separately written exact implementations evaluate the certificate, using moment-curve intersections and the recursive cellular invariant, respectively.

This work has not yet been reviewed by experts. The proofs, the computer checks and the text were produced by AI models (from OpenAI and Anthropic), working under the author's direction.

Keywords: 4-flat graphs, Heawood family, excluded minors, deleted product, van Kampen obstruction.
MSC 2020: 05C10, 57Q35, 05C83.

1 Introduction

For a finite simple graph G, its *full cycle complex* is obtained by attaching a separate closed disk along every simple cycle of G. A graph is *4-flat* if this complex admits a piecewise linear embedding in $\mathbb{R}^4$. Van der Holst introduced this notion [9]; Georgakopoulos and Winter call the complex $X(G)$ the full complex [1, Definition 2.3]. "Full cycle complex" is our descriptive term for the same object. Unless specified otherwise, embeddings below are piecewise linear.

The *Heawood family* consists of the 78 graphs obtained from K_7 and $K_{3,3,1,1}$ by repeated ΔY- and $Y\Delta$-transformations [1, §1.1]. A ΔY-transformation replaces a triangle by a new degree-three vertex joined to its corners. A $Y\Delta$-transformation removes a degree-three vertex and adds the

triangle on its three neighbors; we apply it only when the resulting graph is simple. A minor is obtained by deleting vertices or edges and contracting edges. A class is *minor-closed* if it contains every minor of each of its graphs. An *excluded minor* lies outside such a class while all its proper minors lie inside. Let H be the graph on $\mathbb{Z}/11\mathbb{Z}$ in which vertices are adjacent when their difference is ± 4 or ± 5.

Theorem 1. *The graph H is not 4-flat and has no minor in the Heawood family. Consequently, the Heawood family is not the complete list of excluded minors for 4-flat graphs.*

The graph considered here has vertex set $\{0, \ldots, 10\}$ and edge set

$$E(H) = \{04, 05, 06, 07,\ 15, 16, 17, 18,\ 26, 27, 28, 29,\\ 37, 38, 39, 3\,10,\ 48, 49, 4\,10,\ 59, 5\,10,\ 6\,10\}.$$

Here ij denotes the unordered edge $\{i, j\}$. The graph is 4-regular and isomorphic to the circulant $C_{11}(1, 4)$ under $v \mapsto 3v \pmod{11}$; Figure 1 uses this labeling. Its graph6 string is `J?bFF‘wN?{?`.

Conjecture 1.1 of Georgakopoulos and Winter [1] has two parts:

> Each of the 78 Heawood-family graphs is an excluded minor for 4-flatness. The family contains every excluded minor for 4-flatness.

This is a paraphrase; their second sentence reads: “Even more, there are no further excluded minors.” Van der Holst had proved that all family members are non-4-flat and that K_7 and $K_{3,3,1,1}$ are excluded minors; Georgakopoulos and Winter completed minimality for the other members [1, §1.1, Theorem 6.1]. Theorem 1 refutes completeness. Their footnote 1 attributes the first part explicitly to the end of van der Holst’s §3 [9]. It derives the second part implicitly from van der Holst’s proposed excluded-minor list for $\mu \leq 5$ in §4 and van der Holst’s Conjecture 3, that 4-flatness is equivalent to $\mu \leq 5$; here μ is the Colin de Verdière invariant. Thus the two-part formulation is Georgakopoulos and Winter’s, and completeness combines two conjectures of van der Holst.

Mattman and Pierce found H as the second of their two 4-regular eleven-vertex, twenty-two-edge obstructions to being 2-apex [5, §2], with labels shifted by one. A graph is 2-apex if deleting at most two vertices leaves a planar graph. Kim and Mattman denote H by H_2 [3, §§3.1, 3.3]; their knotless embedding concerns a different property and does not establish 4-flatness. Let G_1 be H with the edge $\{1, 10\}$ added, using the exact labeling of [2]; its graph6 string is `J?bFF‘wNA{?`. It occurs in Mattman and Pavelescu’s ancillary graph6 catalog [4, Appendix A, entry 90 with zero-based indexing] and is isomorphic to the graph denoted G_2 by Kim and Mattman [3, §3]. The author’s earlier paper [2, Theorem 1 and Corollary 3] proved that $\mu(G_1) = 6$, that G_1 has no Heawood-family minor and that the proposed 4-flatness completeness assertion and van der Holst’s conjectured equivalence between 4-flatness and $\mu \leq 5$ cannot both hold. The new step is a certificate on a subcomplex formed from 46 cycle disks of H. Its nonembedding conclusion is independent of $\mu(H)$. The obstruction method is established: van Kampen’s intersection obstruction is described in [7, Appendix D], and the cellular certificate formulation and its relation to van der Holst and Pendavingh’s invariant are explained in [6, Appendix B]. We apply these methods to H, rather than introduce a new obstruction. Combining the certificate with Mattman and Pierce’s computer-search finding that every proper minor of H is 2-apex [5, §2] gives a further consequence.

Corollary 2. *The graph H is an excluded minor for 4-flat graphs. Consequently, G_1 is not 4-flat either.*

Proof. Mattman and Pierce found by computer search that H is minor-minimal not 2-apex [5, §2]; Kim and Mattman recall the result and identify H_2 by graph6 [3, §3.3]. The author’s earlier

computation also established this proper-minor property [2, Lemma 8]. The present public check verifies it directly: each of the 22 edge deletions and 22 edge contractions has a planarizing pair of vertices. Every proper minor is a minor of one of these 44 graphs, since H has no isolated vertices, and 2-apex graphs form a minor-closed class. Thus every proper minor of H is 2-apex.

Every 2-apex graph is 4-flat. Indeed, if deleting two vertices leaves a planar graph Π, the original graph is a subgraph of the join $\Pi * K_2$, obtained by adding two adjacent vertices each adjacent to every vertex of Π. This join is 4-flat by [1, Theorem 5.9(ii)], and its full cycle-complex embedding restricts to each subgraph. Theorem 1 therefore makes H an excluded minor. Finally, G_1 is not 4-flat because its subgraph H is not 4-flat. □

The value of $\mu(H)$ remains undetermined here. Every proper minor is 2-apex and hence has $\mu \leq 5$: planar graphs have $\mu \leq 3$, and adding one vertex increases μ by at most one [8, 1.4(iii), Theorem 2.7]. Deleting any vertex of H gives a 2-apex graph, so the same inequality gives $\mu(H) \leq 6$. If $\mu(H) \leq 5$, then H refutes the proposed equivalence of 4-flatness with $\mu \leq 5$; if $\mu(H) = 6$, then H is also an excluded minor for $\mu \leq 5$. The certificate does not distinguish these alternatives, and Kim and Mattman do not compute this invariant.

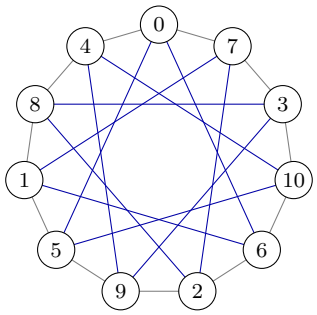


Figure 1: The graph H. The outer cycle has vertex order $0, 4, 8, 1, 5, 9, 2, 6, 10, 3, 7$; the other eleven edges are blue chords.

2 Proof overview

The certificate selects pairs of disjoint cycle disks whose total intersection parity cannot change when the map into $\mathbb{R}^4$ changes. For one explicit map that parity is odd; for an embedding it would be zero. The certificate's zero boundary is what makes the parity invariant.

Attach disks along the 46 simple cycles listed in the certificate, forming a regular cell complex P with one-skeleton H. Appendix A gives the complete lists. The certificate also lists 79 unordered pairs of vertex-disjoint disks. Their sum is a 4-dimensional cycle $\bar{z}$ in the unordered deleted product $Q = D(P)/T$, whose cells are unordered pairs of disjoint cells of P. Here $D(P)$ keeps the pairs ordered and T swaps their factors. Taking both orders gives the symmetric lift z in $D(P)$. The mod-two intersection cochain on Q evaluates to one on $\bar{z}$; equivalently, the cellular invariant $I(z)$ defined in Section 3 is one. The obstruction theorem of van der Holst and Pendavingh then excludes an embedding of P in $\mathbb{R}^4$. An embedding of the full cycle complex would restrict to one of P, so H is not 4-flat.

The absence of Heawood-family minors follows independently from $H \subset G_1$ and the minor exclusion in [2]. The two exact minor checks from that paper are repeated by the accompanying verification.

3 The deleted-product obstruction

All chains in this section have coefficients in $\mathbb{F}_2$. A disk attached along the cycle $(v_0, \dots, v_{r-1})$ has boundary

$$\partial\Delta = \sum_{i=0}^{r-1} \{v_i, v_{i+1}\}, \qquad v_r = v_0;$$

an edge has its two endpoints as boundary, and a vertex has boundary zero. Different disks have distinct open interiors, even if their boundary cycles overlap.

For a regular cell complex P, its *deleted product* $D(P)$ consists of the product cells $a \times b$ for which the closed cells a, b are disjoint. In the complexes used here, this is equivalent to disjointness of their sets of graph vertices. The boundary and the factor-swapping involution T satisfy

$$\partial(a \times b) = \partial a \times b + a \times \partial b, \qquad T(a \times b) = b \times a.$$

In particular, T acts freely on cells and commutes with ∂. A chain z is *symmetric* if $Tz = z$. A *half-chain* of a symmetric chain z contains one product cell from each two-element orbit in its support, so $z = c + Tc$.

An *antipodal* cell complex has a cellular involution taking each closed cell to a disjoint closed cell. The deleted product with factor swapping is such a complex. We use the $\mathbb{Z}_2$-valued invariant I of symmetric cycles defined by van der Holst and Pendavingh [10]. In dimension zero, $I(z)$ is the cardinality modulo two of a half-chain of z. For a symmetric n-cycle z, $n > 0$, and any n-chain c with $z = c + Tc$, their recurrence is

$$I(z) = I(\partial c). \tag{1}$$

This is [10, Lemma 12], using the author-manuscript numbering specified in the bibliography.

The embedding obstruction is [10, Theorem 18]: for a cell complex C with a simplicial underlying space, every symmetric n-cycle z of $D(C)$ has $I(z) = 0$ if and only if there is an even map $C \to \mathbb{R}^n$. An even map is a general-position map for which each pair of disjoint cells of complementary dimensions has zero mod-two intersection number. Only the necessary condition for embedding is used. An open account of this condition and the equality between the crossing cocycle evaluated on a cycle and I is [6, Appendix B.1, pp. 25–26].

Lemma 3. *Let P be obtained from a finite simple graph by attaching separate disks along a finite list of simple cycles. If $D(P)$ contains a symmetric 4-dimensional cycle z with $I(z) = 1$, then P does not embed in $\mathbb{R}^4$.*

Proof. Give each disk its own interior vertex and cone its boundary edges to that vertex. This triangulates P, including when two disk boundaries share several edges. The triangulation supplies the underlying-space hypothesis of [10, Theorem 18]. An embedding would be an even map and hence force $I(z) = 0$, a contradiction.

The conclusion also rules out a topological embedding e. Indeed, there are finitely many pairs of disjoint closed cells, and their images under e are disjoint compact sets. The minimum of their positive separations is positive. A sufficiently close general-position piecewise linear approximation to e therefore keeps every such pair disjoint. It would be an even map, again contradicting [10, Theorem 18]. □

The recurrence also gives a finite way of computing the invariant. Starting with $z_4 = z$, choose a half-chain c_k of z_k and put $z_{k-1} = \partial c_k$ for $k = 4, 3, 2, 1$. Each new chain is symmetric because $\partial c_k + T\partial c_k = \partial z_k = 0$; it is a cycle because $\partial^2 = 0$. Choose also a half-chain c_0 of z_0. Repeated use of [10, Lemma 12], in the form (1), gives

$$I(z) = |c_0| \pmod 2. \tag{2}$$

4 The certificate

The proof will use Lemma 3 by verifying $\partial z = 0$ and $I(z) = 1$. The moment-curve calculation gives a second evaluation of the same invariant. The certificate specifies the edges of H, 46 simple cycles $C_0, \ldots, C_{45}$, and a set S of 79 distinct unordered pairs of indices. Write Δ_i for the disk attached along C_i, and define

$$z = \sum_{\{i,j\} \in S} (\Delta_i \times \Delta_j + \Delta_j \times \Delta_i).$$

For every pair in S, the cycles C_i, C_j are vertex-disjoint. For every edge e and disk Δ_j, the number of selected pairs $\{i, j\}$ for which e belongs to C_i is even. These finite assertions are checked directly and give $\partial z = 0$: they are exactly the coefficients of the edge–disk product cells.

Let $\bar{z} = \sum_{\{i,j\} \in S} [\Delta_i \times \Delta_j]$ in $Q = D(P)/T$. The same boundary calculation gives $\partial \bar{z} = 0$; z is its symmetric lift.

For a general-position map $f : P \to \mathbb{R}^4$, let θ_f assign to an unordered pair of disjoint disks the parity of their intersections. This is a degree-four cochain on Q. Changing f changes θ_f by a coboundary: intersection tracks in a generic homotopy give the corresponding degree-three cochain. This is the mod-two van Kampen obstruction construction; see [7, Appendix D] and its cellular formulation in [6, Appendix B.1]. Consequently $\langle \theta_f, \bar{z} \rangle$ is independent of f, since $\partial \bar{z} = 0$. Its value one certifies a nonzero obstruction class in $H^4(Q; \mathbb{F}_2)$. An embedding has zero intersections and would force this pairing to vanish. The equality of this pairing with $I(z)$ is stated explicitly in [6, Appendix B.1, p. 26], following [10].

There are two independent evaluations of $I(z)$. The cellular evaluation constructs the ordered chain z and applies (1) down through dimensions four, three, two and one. At every dimension it checks symmetry and zero boundary before choosing one cell from each swapping orbit. Equation (2) gives one.

For the intersection evaluation, triangulate each Δ_i by coning its boundary to its private center. Place graph vertex v at parameter $t = v$ and center i at parameter $t = 11 + i$ on the moment curve

$$m(t) = (t, t^2, t^3, t^4).$$

Extend linearly over each triangle. Two triangles with disjoint vertex sets meet precisely when their six parameters, in increasing order, alternate between the triangles. Indeed, every five moment-curve points are affinely independent, and the unique affine dependence of six such points has alternating coefficient signs, by the Vandermonde determinant. Its two sign classes form the Radon partition, giving a single transverse intersection exactly in the alternating case. This is the standard moment-curve test; see [7, Lemma D.2]. A triangle and a disjoint edge cannot meet, again by affine independence of five points. A sufficiently small general-position perturbation therefore preserves these intersection parities.

For each unordered disk pair in S, sum this alternating-order test over its pairs of cone triangles, and then sum over S, modulo two. The pairing identity above identifies the resulting bit with $I(z)$. This second computation also gives one. Both evaluations use integer and mod-two operations; neither involves a floating-point approximation.

Proof of Theorem 1. The finite checks above establish that z is a symmetric 4-dimensional cycle with $I(z) = 1$. Lemma 3 excludes an embedding of P. Since P is a subcomplex of the full cycle complex of H, the graph H is not 4-flat.

The edge list identifies H exactly with $G_1 - \{1, 10\}$ in [2]. That paper's Theorem 1, also checked by the accompanying minor computations, excludes every Heawood-family minor of G_1. Every minor of H is a minor of G_1, so H has no such minor.

Finally, 4-flat graphs form a minor-closed class [1, §1.1], following van der Holst [9]. Choose a minor of H minimal among its non-4-flat minors; this is possible because a finite graph has finitely many minors up to isomorphism. It is an excluded minor for 4-flatness and cannot belong to the Heawood family, completing the proof. □

5 Computational verification

The accompanying data and Appendix A give the cycles and disk pairs explicitly. Each of the two separately written checkers validates the graph, the simplicity of each cycle, the disjointness of each selected pair and the boundary condition. Both read the same graph, cycle list and selected pairs from the JSON certificate. Checker A alone fan-triangulates the disks with private centers and evaluates moment-curve intersections; checker B keeps the disks unsplit and uses their cellular boundaries in the recurrence. The implementations share no checking code. Both accept the certificate for H, as well as certificates for G_1 and the known non-4-flat graph K_7. For each of H and G_1, removing the final listed disk pair gives a nonzero boundary and is rejected by both checkers. A second negative control takes two disjoint copies of the K_7 certificate. Its nonzero chain has zero boundary and invariant $1 + 1 = 0$, so both checkers must reject it at the parity test. The proof needs only the listed subcomplex; completeness of the cycle list is irrelevant.

The minor exclusion uses the two independent implementations published with [2]. One enumerates edge deletions and contractions and tests isomorphism against the Heawood family; the other searches for branch sets. Since G_1 has 23 edges and every Heawood graph has 21 or 22, at most two edge operations can lead to a Heawood minor. Vertex deletions are covered by deleting incident edges and removing isolated vertices. The family is regenerated by closing K_7 and $K_{3,3,1,1}$ under the stated transformations, giving 20 graphs with 21 edges and 58 with 22 edges, as recorded in [5, §2]. The implementations also detect Heawood minors in the supplied positive examples. The public check also makes all 45 direct comparisons: H itself and its 22 single-edge deletions and 22 single-edge contractions. A minor requiring at least two edge operations has at most 20 edges, fewer than any Heawood graph. For each of the 44 proper minors, it also finds a pair of vertices whose deletion leaves a planar graph, independently verifying the proper-minor input to Corollary 2.

The complete verification takes less than a minute on one CPU core. The accompanying README gives the dependencies and reproduction command. The finite certificates and their arithmetic are checked by these programs; the topological implication relies on the two cited inputs from van der Holst and Pendavingh and their intersection interpretation, also presented in [6, Appendix B]. The proof is not formalized in a proof assistant.

Appendix A The explicit certificate

The following lists completely specify the subcomplex P and the cycle $\bar{z} = \sum_{(i,j)\in S}[\Delta_i \times \Delta_j]$ in the unordered deleted product. A row $(v_0, \ldots, v_r)$ denotes the closed cycle with successive edges $v_0v_1, \ldots, v_{r-1}v_r, v_rv_0$. All indices start at zero.

i	C_i	i	C_i
0	$(0,4,8,1,5)$	23	$(1,5,9,3,7)$
1	$(0,4,8,1,7)$	24	$(1,5,9,4,8)$
2	$(0,4,8,2,6)$	25	$(1,5,10,3,7)$
3	$(0,4,8,3,7)$	26	$(1,5,10,3,8)$
4	$(0,4,9,2,6)$	27	$(1,5,10,6)$
5	$(0,4,9,2,7)$	28	$(1,6,2,7)$
6	$(0,4,9,3,10,5)$	29	$(1,6,2,8)$
7	$(0,4,9,5)$	30	$(1,6,10,3,8)$
8	$(0,4,10,3,7)$	31	$(1,6,10,4,8)$
9	$(0,4,10,3,9,5)$	32	$(1,7,2,8)$
10	$(0,4,10,5)$	33	$(1,7,3,8)$
11	$(0,4,10,5,1,6)$	34	$(2,6,10,3,7)$
12	$(0,4,10,6)$	35	$(2,6,10,3,9)$
13	$(0,5,1,6)$	36	$(2,6,10,4,8)$
14	$(0,5,1,7)$	37	$(2,6,10,5,9)$
15	$(0,5,9,2,7)$	38	$(2,7,3,8)$
16	$(0,5,9,3,7)$	39	$(2,7,3,8,4,9)$
17	$(0,5,10,6)$	40	$(2,7,3,9)$
18	$(0,6,1,7)$	41	$(2,8,3,9)$
19	$(0,6,2,7)$	42	$(2,8,4,9)$
20	$(0,6,10,3,7)$	43	$(3,8,4,9,5,10)$
21	$(1,5,9,2,6)$	44	$(3,8,4,10)$
22	$(1,5,9,2,8)$	45	$(3,9,5,10)$

For every graph edge e and every disk index j, count the selected pairs containing j whose other cycle contains e. These are the coefficients of $\partial\bar{z}$; every count is even. The nonzero integer counts have the following multiplicities: 2 occurs 340 times, 4 occurs 12 times. Thus $\partial\bar{z} = 0$; equivalently the ordered lift z is a cycle.

The table below lists every pair in S and its moment-curve intersection coefficient θ_{ij}. For a direct check, cone C_i to parameter $11 + i$ and order each six-element set of two disjoint triangles. A triangle pair contributes one exactly when its two triples alternate in that order. Sum these contributions for each disk pair and reduce modulo two.

i	j	θ_{ij}	i	j	θ_{ij}	i	j	θ_{ij}
0	34	1	8	22	1	16	36	0
0	35	1	8	29	0	17	32	0
0	40	0	9	28	0	17	33	0
1	35	1	9	29	0	17	39	0
1	37	1	9	32	0	17	42	0
1	45	0	10	38	0	18	41	0
2	23	0	10	40	0	18	42	0
2	25	0	10	41	0	18	43	0
2	45	0	11	38	0	18	45	0
3	21	1	11	40	0	19	24	0
3	27	0	11	41	0	19	26	0
3	37	1	12	22	0	19	43	0
4	25	0	12	23	0	20	22	1
4	26	0	12	32	0	20	24	1
4	33	0	12	40	0	20	42	0
5	26	0	13	38	0	21	44	0
5	27	0	13	39	0	23	36	0
5	30	0	13	42	0	24	34	1
6	28	0	14	35	0	25	42	0
6	29	0	14	36	0	27	39	0
6	32	0	14	41	0	28	43	0
7	29	0	14	44	0	28	44	0
7	30	0	15	30	0	29	45	0
7	32	0	15	31	0	31	40	0
7	33	0	15	44	0	33	37	0
7	34	0	16	29	0			
8	21	1	16	31	0			

Exactly 11 of the 79 listed coefficients are one. Hence $\langle \theta, \bar{z} \rangle = 1$ in $\mathbf{F}_2$. The ordered-lift half-chain calculation independently gives $I(z) = 1$. The public check regenerates this appendix from `data/h.json` and requires an exact match; both obstruction checkers independently verify the underlying certificate.

References

[1] A. Georgakopoulos and M. Winter, On 2-complexes embeddable in 4-space, and the excluded minors of their underlying graphs, arXiv:2408.12681v1 (2024).

[2] P. Graham, The Heawood family is not the complete list of excluded minors for Colin de Verdière invariant at most 5, *heawood-family-mu5*, version 1 (2026), doi:10.5281/zenodo.24128771.

[3] H. Kim and T. W. Mattman, Dips at small sizes for topological graph obstruction sets, *Discrete Appl. Math.* 360 (2025), 139–166, doi:10.1016/j.dam.2024.08.022. Section numbers refer to arXiv:2205.14255v2.

[4] T. W. Mattman and A. Pavelescu, Three thousand obstructions to knotless embedding, arXiv:2607.01414v1 (2026).

[5] T. W. Mattman and M. Pierce, The K_{n+5} and $K_{3^2,1^n}$ families and obstructions to n-apex, in *Knots, Links, Spatial Graphs, and Algebraic Invariants*, Contemp. Math. 689,

Amer. Math. Soc., 2017, 137–158, doi:10.1090/conm/689/13856. Section numbers refer to arXiv:1603.00885v1.

[6] V. Kaluža and M. Tancer, Even maps, the Colin de Verdière number and representations of graphs, arXiv:1907.05055v2 (2020). All page and proposition references use this version.

[7] J. Matoušek, M. Tancer and U. Wagner, Hardness of embedding simplicial complexes in $\mathbb{R}^d$, *J. Eur. Math. Soc.* 13 (2011), 259–295, doi:10.4171/JEMS/252.

[8] H. van der Holst, L. Lovász and A. Schrijver, The Colin de Verdière graph parameter, in *Graph Theory and Combinatorial Biology*, Bolyai Soc. Math. Stud. 7 (1999), 29–85, open text.

[9] H. van der Holst, Graphs and obstructions in four dimensions, *J. Combin. Theory Ser. B* 96 (2006), 388–404, doi:10.1016/j.jctb.2005.09.004.

[10] H. van der Holst and R. A. Pendavingh, On a graph property generalizing planarity and flatness, *Combinatorica* 29 (2009), 337–361, doi:10.1007/s00493-009-2219-6. Lemma 12, Theorem 18 and section numbers refer to the author manuscript dated April 26, 2008.