A conjecture of Hopkins on golden Rado numbers

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Abstract

A 2-coloring of $\{1,\dots,N\}$ satisfies the golden mean shift if no two consecutive integers both receive color 1. For $1 \le a \le b$, Hopkins defined the golden Rado number $R_0^{\varphi}(a,b)$ as the least N such that every such coloring with 1 colored 0 contains a monochromatic solution of $ax+by=z$, and conjectured that $R_0^{\varphi}(a,\ell a) = a^2(\ell+1)^2$ for all positive integers a and ℓ. Hopkins proved the case $\ell = 1$, and determined $R_0^{\varphi}(1,b)$ for every b; that evaluation shows the formula fails when $a = 1$ and $\ell \ge 2$. We prove the formula for every $a \ge 2$ and $\ell \ge 1$, which settles all the remaining cases of the conjecture. The lower bound is the coloring that gives color 1 exactly to the multiples of a that are at least $a(\ell+1)$. The upper bound is a forcing argument: a descent in steps of ℓ shows that a residue $r \le \min(a,\ell)$ has color 1, and two families of solutions then make color 1 invariant under shifts by ℓ on an initial interval, which leads to a contradiction. The proof is elementary, and the theorem, both bounds included, is checked in the Lean proof assistant from the definitions.

This work has not yet been reviewed by experts. The proofs, the Lean formalization and the text were produced by AI models (from OpenAI and Anthropic), working under the author's direction.

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1 Introduction

1.1 Main result

A *golden coloring* of $[1,N] = \{1,\dots,N\}$ is a map $\Delta : [1,N] \to \{0,1\}$ with $\Delta(1) = 0$ and no i with $\Delta(i) = \Delta(i+1) = 1$. For positive integers a, b, a solution (x,y,z) of $ax+by=z$ in $[1,N]$ is *monochromatic* if $\Delta(x) = \Delta(y) = \Delta(z)$; the entries x and y may be equal. Following Hopkins [2, Definition 2], for $1 \le a \le b$ the golden Rado number $R_0^{\varphi}(a,b)$ is the least positive integer N such that every golden coloring of $[1,N]$ contains a monochromatic solution of $ax+by=z$. (Hopkins writes R_0^S for a binary shift S and R_0^{φ} for the golden mean shift; the subscript records $\Delta(1) = 0$.) For example, $R_0^{\varphi}(1,1) = 4$ [2, Section 2]. Suppose a golden coloring of $[1,4]$ has no monochromatic solution of $x+y=z$. Then $\Delta(2) = 1$ because of $(1,1,2)$, so $\Delta(3) = 0$, and either color of 4 completes $(1,3,4)$ or $(2,2,4)$, a contradiction; the coloring $0,1,0$ of $[1,3]$ avoids all solutions.

Theorem 1. *For all integers $a \ge 2$ and $\ell \ge 1$,*

$$R_0^{\varphi}(a,\ell a) = a^2(\ell+1)^2.$$

Hopkins stated the formula as follows [2, Conjecture 1]: "Given positive integers a and ℓ, the golden Rado number for the case $\Delta(1) = 0$ is $R_0^{\varphi}(a, \ell a) = (\ell+1)^2 a^2$." The case $\ell = 1$ is Hopkins's Theorem 2, $R_0^{\varphi}(a, a) = 4a^2$ [2, Theorem 2]. For $a = 1$, Hopkins's Theorem 1 gives $R_0^{\varphi}(1, b) = 2b+2$ for odd b and $R_0^{\varphi}(1, b) = 2b$ for even b [2, Theorem 1]; this equals $(b+1)^2$ only for $b = 1$, so the conjectured formula holds at $a = 1$ only when $\ell = 1$. Theorem 1 proves it in every other case.

1.2 Background

Without the coloring constraint, the 2-color Rado number of $ax+by = z$ for $1 \le a \le b$ is $a(a+b)^2+b$, as conjectured by Hopkins and Schaal [3] and proved by Guo and Sun [1] (we quote the formula as stated in [2, equation (1)]). Hopkins introduced shift-constrained Rado numbers in [2], where the colorings are required to be words of a binary shift from symbolic dynamics [6]; the golden mean shift forbids two consecutive ones. Since the color of 1 can no longer be fixed by symmetry, Hopkins treats $\Delta(1) = 0$ and $\Delta(1) = 1$ separately and writes that "a full understanding of $R_0^{\varphi}(a, b)$, $R_1^{\varphi}(a, b)$ is desired" [2, Section 5]. Hopkins supported Conjecture 1 with computations and acknowledges earlier exploration of these topics by Katrina Luckenbach and Matthew Vieira as undergraduate research students [2, p. 13]. Background on Rado numbers is in Landman and Robertson's book [4, Chapter 9].

1.3 What is new

Hopkins's proof of the case $\ell = 1$ and the descent below both start from the solution $(1, 1, a + a\ell)$. For $\ell = 1$ a single descent already reaches a contradiction. For $\ell \ge 2$ the descent stops at a residue $r \ge 2$, and the new step is the second stage: two families of solutions that make color 1 invariant under shifts by ℓ on the interval $[1, a\ell(r+1)]$. The lower-bound coloring is the same for all ℓ, and it is the hypothesis $a \ge 2$ that makes it a golden coloring.

2 Proof overview

Write $s = a(\ell+1)$, so the claimed value is s^2, and let $b = a\ell$. Every positive solution of $ax+by = z$ has $z = a(x+\ell y)$, a multiple of a that is at least s. Coloring exactly these numbers with 1 forces any monochromatic solution to have $x, y \ge s$, hence $z \ge s^2$; this is the lower bound (Section 3).

For the upper bound (Section 4) take a golden coloring of $[1, s^2]$ without monochromatic solutions. Because 1 has color 0, the solution $(1, 1, s)$ gives s color 1. The identity $a(t-\ell)+b(t+1) = st = at+bt$ moves color 1 from t down to $t-\ell$, so color 1 descends from s to its least positive residue r modulo ℓ. Then $r \ge 2$, and both neighbors of r have color 0. Solutions with $y \in \{r-1, r, r+1\}$ propagate color 1 by $\pm\ell$ on $[1, T]$, where $T = b(r+1)$. Whatever the color of ℓ, propagation contradicts one of the solutions $(\ell, 1, 2b)$ and (ℓ, r, T). All entries stay inside $[1, s^2]$ because $r \le \min(a, \ell)$.

3 The lower bound

Lemma 2. *Let $a \ge 2$, $\ell \ge 1$ and $s = a(\ell+1)$. Color t with 1 exactly when a divides t and $t \ge s$. For every M with $1 \le M < s^2$, the restriction to $[1, M]$ is a golden coloring with no monochromatic solution of $ax + a\ell y = z$.*

Proof. Since $a \ge 2$, the integer 1 is not a multiple of a, and two consecutive integers are never both multiples of a; so the coloring is golden. Let (x, y, z) be a solution with entries in $[1, M]$. Then

$z = a(x + \ell y)$ is a multiple of a and $z \geq a(1 + \ell) = s$, so z has color 1. If x and y also had color 1, then $x, y \geq s$ and $z \geq a(s + \ell s) = s^2 > M$, which is impossible. □

For $a = 1$ every integer $t \geq s$ receives color 1, so the coloring is not golden; this is where the hypothesis $a \geq 2$ enters.

4 The upper bound

Fix $a, \ell \geq 1$, put $s = a(\ell + 1)$ and $b = a\ell$, and let Δ be a golden coloring of $[1, s^2]$ with no monochromatic solution of $ax + by = z$. We derive a contradiction, which proves that every golden coloring of $[1, s^2]$ contains a monochromatic solution. Two facts are used throughout. If (x, y, z) is a solution in $[1, s^2]$ and $\Delta(x) = \Delta(y)$, then $\Delta(z) \neq \Delta(x)$; and if $\Delta(y) = \Delta(z)$, then $\Delta(x) \neq \Delta(y)$. Also, a number of color 1 has both neighbors in $[1, s^2]$ of color 0.

Let r be the integer with $1 \leq r \leq \ell$ and $r \equiv a \pmod{\ell}$, that is, $r = ((a - 1) \bmod \ell) + 1$. Then $r \leq a$, so $r \leq \min(a, \ell)$, and $s = a + b \equiv a \equiv r \pmod{\ell}$.

Lemma 3 (Descent)**.** $\Delta(s) = 1$*, and* $\Delta(t) = 1$ *with* $\ell < t \leq s$ *implies* $\Delta(t - \ell) = 1$*. Consequently* $\Delta(r) = 1$*,* $r \geq 2$ *(so* $\ell \geq 2$*), and* $\Delta(r - 1) = \Delta(r + 1) = 0$*.*

Proof. The solution $(1, 1, s)$ has $\Delta(1) = 0$, so $\Delta(s) = 1$. Let $\ell < t \leq s$ with $\Delta(t) = 1$. The solution (t, t, st), with $st \leq s^2$, gives $\Delta(st) = 0$, and $\Delta(t + 1) = 0$ because the coloring is golden. Since

$$a(t - \ell) + b(t + 1) = at + bt = st,$$

the solution $(t - \ell, t + 1, st)$ has $\Delta(t + 1) = \Delta(st) = 0$, so $\Delta(t - \ell) = 1$. Starting from s and stepping down by ℓ while the current value exceeds ℓ, we stay in the residue class of s modulo ℓ and stop at r; hence $\Delta(r) = 1$. As $\Delta(1) = 0$, we get $r \geq 2$, and the golden condition gives $\Delta(r \pm 1) = 0$. □

In particular, for $\ell = 1$ we have $r = 1$, and Lemma 3 is already a contradiction. From now on $\ell \geq 2$. Put

$$T = b(r + 1) = a\ell(r + 1), \qquad K = a(\ell + 1)^2,$$

so that $aK = s^2$. Since $r \leq \min(a, \ell)$,

$$T + \ell r = a\ell r + a\ell + \ell r \leq a\ell^2 + 2a\ell < K, \qquad 3b \leq T. \tag{1}$$

Lemma 4 (Shifts)**.** *Let* $1 \leq t$ *and* $t + \ell \leq T$*. Then* $\Delta(t) = 1$ *if and only if* $\Delta(t + \ell) = 1$*.*

Proof. Let $u \leq T$ be a positive integer with $\Delta(u) = 1$. The solution $(u, r, a(u + \ell r))$ has both inputs of color 1 by Lemma 3, and its output is at most $a(T + \ell r) < aK = s^2$ by (1); so $\Delta(a(u + \ell r)) = 0$.

If $\Delta(t) = 1$, apply this with $u = t$. The identity $a(t + \ell) + b(r - 1) = a(t + \ell r)$ gives the solution $(t + \ell, r - 1, a(t + \ell r))$, whose last two entries have color 0 (the middle one by Lemma 3, and $r - 1 \geq 1$). Hence $\Delta(t + \ell) = 1$.

If $\Delta(t + \ell) = 1$, apply it with $u = t + \ell$: the output $a(t + \ell + \ell r) = at + b(r + 1)$ has color 0. The solution $(t, r + 1, at + b(r + 1))$ has its last two entries of color 0, so $\Delta(t) = 1$. □

Proof of Theorem 1. By Lemma 2, no positive $M < s^2$ has the forcing property, so $R_0^{\varphi}(a, \ell a) \geq s^2$. For the upper bound, suppose Δ is as above, with $\ell \geq 2$. The multiples $\ell, 2\ell, \ldots, T$ of ℓ all lie in $[1, T]$, so by Lemma 4 they all have the same color.

If $\Delta(\ell) = 0$, the solution $(\ell, 1, 2b)$ has $\Delta(\ell) = \Delta(1) = 0$, so the multiple $2b = 2a\ell \leq T$ of ℓ has color 1, a contradiction. If $\Delta(\ell) = 1$, then $\Delta(T) = 1$; but $a\ell + br = b(r + 1) = T$, so (ℓ, r, T) is a solution with $\Delta(\ell) = \Delta(r) = 1$ and therefore $\Delta(T) = 0$, again a contradiction. Thus every golden coloring of $[1, s^2]$ contains a monochromatic solution, and $R_0^{\varphi}(a, \ell a) = s^2 = a^2(\ell + 1)^2$. □

Remark 5. The upper-bound argument does not use $a \geq 2$: for every $a, \ell \geq 1$, each golden coloring of $[1, a^2(\ell+1)^2]$ contains a monochromatic solution of $ax + a\ell y = z$. For $a = 1$ and $\ell \geq 2$ this bound is not sharp, by Hopkins's Theorem 1.

5 Formal verification

The theorem is proved in Lean 4 [5] with Mathlib [7], from the following definitions (the file `Hopkins/Definitions.lean`). A coloring is a function $c : \mathbb{N} \to \{\text{false}, \text{true}\}$, with true playing the role of color 1; only its values on $[1, N]$ matter.

```
def Valid (N : ℕ) (c : ℕ → Bool) : Prop :=
  c 1 = false ∧ ∀ i : ℕ, 1 ≤ i → i + 1 ≤ N →
    ¬ (c i = true ∧ c (i + 1) = true)

def Mono (a b N : ℕ) (c : ℕ → Bool) : Prop :=
  ∃ x y z : ℕ, 1 ≤ x ∧ x ≤ N ∧ 1 ≤ y ∧ y ≤ N ∧
    1 ≤ z ∧ z ≤ N ∧ a * x + b * y = z ∧ c x = c y ∧ c y = c z

def Forces (a b N : ℕ) : Prop :=
  ∀ c : ℕ → Bool, Valid N c → Mono a b N c

def ForcingSet (a b : ℕ) : Set ℕ := {N | 1 ≤ N ∧ Forces a b N}

noncomputable def R0phi (a b : ℕ) : ℕ := sInf (ForcingSet a b)
```

Thus `Valid N c` says that c restricts to a golden coloring of $[1, N]$, `Mono` asks for a monochromatic solution with all entries in $[1, N]$ (with $x = y$ allowed), and `R0phi a b` is the least positive N with the forcing property, which is Hopkins's $R_0^{\varphi}(a, b)$. The main theorems are

```
theorem Hopkins.formula_isLeast (a ell : ℕ) (ha : 2 ≤ a) (hell : 1 ≤ ell) :
    IsLeast (ForcingSet a (a * ell)) (a ^ 2 * (ell + 1) ^ 2)
theorem Hopkins.formula (a ell : ℕ) (ha : 2 ≤ a) (hell : 1 ≤ ell) :
    R0phi a (a * ell) = a ^ 2 * (ell + 1) ^ 2
```

The first says that $a^2(\ell+1)^2$ has the forcing property and that no smaller positive integer does, which is Theorem 1; the second deduces the value of the infimum from it, so no convention about the infimum of an empty set is involved. The development also proves the two bounds separately: `Hopkins.upper` (forcing at $a^2(\ell+1)^2$ for all $a, \ell \geq 1$) and `Hopkins.lower` (Lemma 2, with the coloring `lowerColor`). As checks on the definitions, it proves Hopkins's values $R_0^{\varphi}(1,1) = 4$ and $R_0^{\varphi}(1,2) = 4$ from Hopkins's Theorem 1, the second showing the failure of the printed formula at $a = 1$, $\ell = 2$.

The proof uses only Lean's standard logical axioms, with no added axioms and no compiled evaluation. The README in the `lean` folder accompanying this paper gives the versions, the commands that reproduce the check and its axiom report.

References

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