Greedy play can require more colors in the coloring game

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Abstract

In the coloring game, Alice and Bob alternately color vertices properly from a fixed palette, with Alice trying to color the whole graph. The game chromatic number is the least palette size for which she can win. In the greedy game the players choose vertices, each receiving the least available positive integer; Alice minimizes and Bob maximizes the number of colors used. Havet and Zhu asked in June 2011 whether the resulting game Grundy number is always at most the game chromatic number. We give an explicit finite simple graph G for which $\chi_g(G) = 5$ and $\Gamma_g(G) = 6$, answering the first part of their question negatively. The proof consists of finite strategy certificates checked by two separately written certificate checkers for the two unrestricted Alice-first games, together with a clique obstruction and a short greedy strategy. It does not establish arbitrarily large separation between the parameters.

This work has not yet been reviewed by experts. The proofs, the search code, the certificate checkers and the text were produced by AI models (from OpenAI and Anthropic), working under the author's direction.

Computer-assisted.

An outside AI model suggested the clique and degree observations used for the exact values.

1 Introduction

The ordinary coloring game is played on a finite simple graph with a palette of k colors. Alice and Bob alternately give an uncolored vertex a color different from those of its colored neighbors. Alice wins if the whole graph is colored; otherwise Bob wins. The least k for which Alice has a winning strategy is the *game chromatic number* $\chi_g(G)$. In the greedy game, a chosen vertex receives the least positive integer absent from its colored neighbors. Alice minimizes and Bob maximizes the largest color used. The value is the *game Grundy number* $\Gamma_g(G)$. Alice moves first in both games, either player may choose any uncolored vertex, and neither player may pass.

Theorem 1. *The graph G defined in Section 3 is a finite simple graph with 11 vertices and 27 edges satisfying*

$$\chi_g(G) = 5, \qquad \Gamma_g(G) = 6.$$

Both values concern unrestricted vertex selection with Alice moving first, alternating turns, and no passes.

Havet and Zhu introduced the game Grundy number in a report first posted on June 15, 2011. The first sentence of Problem 5 in version 1 reads:

> "Is it true that $\Gamma_g(G) \leq \chi_g(G)$ for all graph G?"

The problem then asks separately whether the game Grundy number can be arbitrarily large compared with the game chromatic number [1, p. 6]. Theorem 1 answers the first part negatively. It does not answer the second part, and makes no assertion about the smallest possible counterexample.

Hollom restated the inequality as Question 3.4 and explained an obstacle to transferring an ordinary winning strategy to greedy play [2]. Hollom's ordered-game example concerns a prescribed sequence of vertex locations. It does not itself give an unrestricted counterexample. The new contribution here is an explicit graph together with an ordinary winning strategy for Alice and a greedy strategy for Bob that survives every legal choice of vertex by Alice. This universal coverage of Alice's choices is the essential additional obligation beyond an ordered example; it is supplied by the finite certificates.

There is also a distinct question in the reverse direction. Krawczyk and Walczak proved that the game chromatic number is not bounded by a function of the ordinary, static Grundy number [3, Theorem 7]. The static Grundy number is the maximum number of colors used by greedy coloring over all vertex orders, so it is at least $\Gamma_g(G)$. Their theorem therefore also rules out bounding $\chi_g(G)$ by a function of $\Gamma_g(G)$. This earlier result concerns the reverse boundedness question, rather than the inequality answered here. The present proof uses the definitions and question of Havet and Zhu and the distinction emphasized by Hollom; it does not use the construction of Krawczyk and Walczak.

Nearby work studies connected greedy coloring (Lima, Marcilon, Martins and Sampaio [4]; Marcilon and Ribeiro [5]), ordinary coloring on complete multipartite graphs (Furtado, Dantas, de Figueiredo and Del-Rio-Palma [6]), majority coloring games (Chawda, Nanoti, Sankarnarayanan and Srinivasan [7]), connected games on corona products (Mesarič Štesl [8]), and the difference from the ordinary chromatic number (Hollom [9]). Costa, Pessoa, Sampaio and Soares [10] and Marcilon, Martins and Sampaio [11] study computational complexity; Figueiredo and dos Santos [12] study two-turn asymmetric games. Nothing from these nearby works is used here.

Section 2 describes the proof and the finite games used in the certificates. Section 3 specifies the graph. Section 4 gives the certificate argument, and Section 5 describes the reproducible checks.

2 Games and proof overview

The proof compares two games on the same graph with the same five colors; it does not compare separate graph families or rely on changing the palette of an ordinary strategy.

For example, if a vertex has colored neighbors using color 1 and no other color, an ordinary move with five colors may give it any of $2, 3, 4, 5$. The greedy move must give it color 2. Although greedy moves form a subset of ordinary moves at a fixed position, both players are subject to this restriction. An ordinary winning strategy for Alice may ask her to make a move that the greedy rule forbids.

Write $V(G)$ for the vertex set and $N_G(v)$ for the neighbors of a vertex v. A palette-k position is a function

$$s : V(G) \longrightarrow \{0, 1, \ldots, k\},$$

where 0 denotes an uncolored vertex and adjacent nonzero values must differ. For an uncolored vertex v, define its available colors by

$$A_k(s, v) = \{1, \ldots, k\} \setminus \{s(u) : u \in N_G(v),\ s(u) \neq 0\}.$$

An ordinary legal move chooses any uncolored v and any $c \in A_k(s, v)$. A greedy legal move in the palette-k game chooses any uncolored v with $A_k(s, v) \neq \varnothing$ and uses $c = \min A_k(s, v)$. In each case the new position changes only $s(v)$, from 0 to c.

The certificates stop a palette-k game as soon as either all vertices are colored, an Alice win, or an uncolored vertex has no available color, a Bob win. Such a vertex is called *blocked*. An ordinary game may instead be described as continuing until no legal move remains. These conventions have the same winner: once a vertex is blocked, its neighbors' colors never disappear, and a full coloring is impossible.

The unrestricted greedy game has no fixed palette. A selected vertex always receives the least positive integer absent from its colored neighbors, so a move is possible until every vertex is colored. Its used colors form an initial interval of the positive integers: a first use of color c requires colored neighbors of every smaller positive color. Consequently its largest color and its number of colors agree.

The two original certificates give the outer bounds. The ordinary certificate selects a legal move for Alice and covers every legal move by Bob. It forces a full proper coloring from the five-color palette. The greedy certificate selects a legal move for Bob and covers every greedy move by Alice. It forces a blocked vertex in the palette-five game. Since the five colors on that vertex's neighbors persist, unrestricted greedy play must eventually use a color at least six. A five-vertex clique supplies the matching ordinary lower bound. For the greedy upper bound, Alice ensures that the only two vertices of degree greater than five are colored within the first three moves. Every other vertex then receives a color at most six.

Each certificate is a directed graph of literal labeled positions. Different histories reaching the same position share its continuation. There is no quotient by graph automorphisms or by color permutations. Every arc colors one additional vertex, so correctness reduces to a finite induction. The argument establishes two specific strategies; it requires neither a search over all graphs nor an exhaustive solution of every position in either game.

3 The graph

The following list specifies the example independently of its drawing. Its vertices are $0, 1, \ldots, 10$. In the right column of Table 1, each entry v in row u denotes the edge $\{u, v\}$; only neighbors with larger labels are listed. Thus each edge appears exactly once, and there are no additional edges.

u	Neighbors greater than u	u	Neighbors greater than u
0	$3, 5, 7, 8$	6	$9, 10$
1	$4, 6, 8, 9, 10$	7	9
2	$5, 6, 7, 9, 10$	8	10
3	7	9	10
4	$6, 8, 9, 10$	10	$\varnothing$
5	$7, 8, 10$		

Table 1: The complete edge list of G, grouped by smaller endpoint.

The machine-readable graph is `data/graph.json`. All three strategy certificates contain the same vertex set and edge list, and the public entry point checks their equality to this graph before checking either strategy. It binds the original pair to palette size five and the new greedy certificate to palette size six, all with Alice-first play and unrestricted vertex selection. The ordinary certificate's first move colors vertex 5 with color 1; subsequent choices are given by its positions and arcs, rather than by a short case-based strategy in the text.

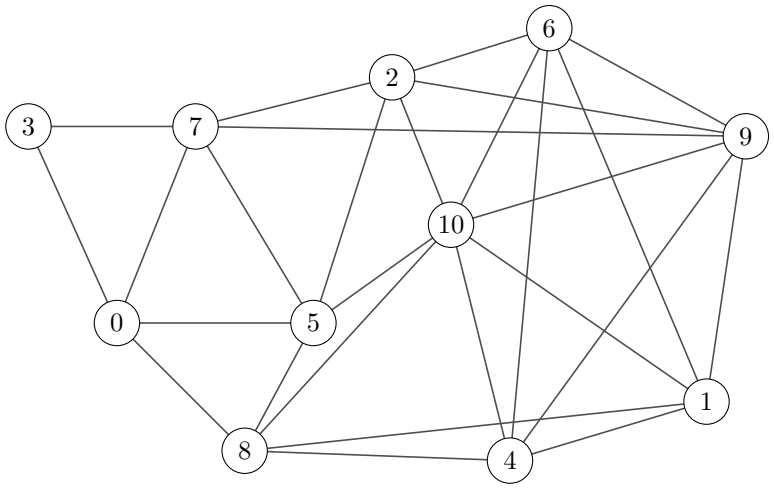


Figure 1: The graph G. Crossings in the drawing are not vertices. The drawing is generated from the edge list in `data/graph.json`.

4 Why the certificates prove the theorem

This section gives the finite induction behind the checks and explains why the greedy certificate implies a lower bound in the unrestricted game.

4.1 Local strategy obligations

A strategy certificate declares a game, a starting player, a palette size, a graph and a claimed winner. Its nodes are proper partial colorings. The root is the all-zero position. A node lists moves (v, c), each leading to the position obtained by assigning color c to the previously uncolored vertex v. Alice moves when the number of colored vertices is even; Bob moves when it is odd.

At every reachable node the checker recomputes all legal moves directly from the graph and the partial coloring. A terminal node lists no moves, and its recomputed winner must be the declared winner. At a nonterminal node belonging to the declared winner, the certificate lists exactly one legal move. At a nonterminal node belonging to the other player, it lists exactly all legal moves, without duplication. Every listed successor must be present, and all nodes must be reachable from the root.

Lemma 1. *Consider either palette-k game defined in Section 2, with Alice first and with stopping at a blocked vertex. If a finite certificate satisfies the local strategy obligations above, its declared winner has a winning strategy from the all-zero position.*

Proof. For each certificate position s, induct on the number of its uncolored vertices. A terminal position has the declared winner by the terminal check. At a nonterminal position where that winner moves, the selected legal move leads to a position with one fewer uncolored vertex, which is winning by induction. At a position where the opponent moves, all legal moves are included, and every successor is winning by induction. This proves the claim at every reachable position, including the root. The strict increase in the number of colored vertices along each arc also rules out cycles. □

4.2 From a blocked vertex to a sixth greedy color

The lower bound uses a threshold certificate, so its connection with the unrestricted greedy game needs a separate argument.

Lemma 2. *Let H be a finite simple graph. Suppose Bob has a winning strategy in the Alice-first greedy palette-k game of Section 2, with unrestricted vertex selection, alternating turns, no passes, and stopping when a vertex is blocked. Then Bob can force a color greater than k in the unrestricted greedy game on H. In particular, $\Gamma_g(H) > k$.*

Proof. Until a blocked vertex occurs, every uncolored vertex has some available color in $\{1, \dots, k\}$. Its least available positive integer is therefore the same in the unrestricted game and in the palette-k game. Bob follows his palette-k strategy during this common part of play. Every choice of vertex by Alice is a legal choice in the threshold game and is covered by that strategy.

Bob's winning strategy reaches a blocked vertex v before a full coloring. At that point every color in $\{1, \dots, k\}$ occurs on a colored neighbor of v. These colors persist. The unrestricted game continues until all vertices are colored, since each turn colors one previously uncolored vertex and no player may pass. When v is colored, its greedy color is consequently greater than k. Thus Bob forces the required lower bound, regardless of subsequent moves. □

4.3 The matching local bounds

The remaining bounds use only the labeled graph and a short opening strategy.

Lemma 3. *The graph G satisfies $\chi_g(G) \geq 5$ and $\Gamma_g(G) \leq 6$.*

Proof. All ten pairs of vertices in $\{1, 4, 6, 9, 10\}$ are edges in Table 1. Every complete proper coloring therefore uses at least five colors, so Alice cannot win with any smaller palette.

Every vertex outside $\{9, 10\}$ has degree at most five. A vertex of degree d receives a greedy color at most $d + 1$: a larger color would require distinct colored neighbors of every smaller positive color. Alice first chooses vertex 10, which receives color 1. If Bob chooses vertex 9 on move two, its color is at most 2. Otherwise Alice chooses 9 on move three, when its color is at most 3 because only two vertices have been colored. In either branch, both exceptional vertices are colored by move three. All later choices are among vertices of degree at most five and receive colors at most six. Alice may choose any remaining vertex thereafter. Each move colors a new vertex, so play finishes after eleven moves, with at most six distinct colors, for every legal Bob response. □

Proof of Theorem 1. Table 1 specifies a simple graph of the stated order and size. The two checks described in Section 5 accept an ordinary certificate with declared winner Alice and a greedy certificate with palette $k = 5$ and declared winner Bob for this graph and the stated conventions.

By Lemma 1, Alice has a winning ordinary strategy with five colors. Hence $\chi_g(G) \leq 5$ directly from the definition as the least winning palette size. No assertion that ordinary winning palettes are monotone is needed. The same lemma gives Bob a winning strategy in the greedy palette-five game. Lemma 2 then gives $\Gamma_g(G) > 5$. Combining these outer bounds with Lemma 3 proves the two exact values. □

5 Finite verification and reproducibility

The bundle contains the graph, the three finite strategies and two separately written certificate checkers. Checking requires Python 3 and its standard library, with no graph generator, search

executable or external mathematical package. From the bundle's top-level directory, the command is

```
python3 -I checks/verify.py
```

The same command checks all three certificates, the local graph facts, and the corruption tests. Detailed format and checker documentation is in `checks/README.md`.

5.1 Data and checker independence

The files `data/ordinary-k5.json.gz`, `data/greedy-k5.json.gz` and `data/greedy-k6.json.gz` are gzip-compressed JSON objects. Besides the game conventions and graph, each contains a map from comma-separated color vectors to lists of moves. The coordinate order is the vertex order $0, \ldots, 10$; 0 is the uncolored marker. All states and moves are labeled literally. A strategy can discard unneeded choices at its winner's turns, while retaining every legal choice at the opponent's turns. The resulting sizes are reported in Table 2; these are certificate sizes, not full game-tree sizes.

Game	Declared winner	Nodes	Arcs	Terminal nodes
Ordinary, $k = 5$	Alice	108 669	216 995	2 800
Greedy, $k = 5$	Bob	5 594	6 730	2 010
Greedy, $k = 6$	Alice	2 223	3 339	208

Table 2: The finite strategy certificates.

The programs `checks/check.py` and `checks/check_independent.py` implement the local obligations of Section 4 separately. Neither imports the search program or the other checker. They reconstruct adjacency, properness, available colors, the player to move, legal moves and terminal outcomes from each certificate. In particular, terminal winners are recomputed; they are not trusted labels attached to leaves. The entry point also checks the graph and game bindings against the public statement's five-color outer bounds and six-color upper bound. Thus acceptance of a different graph, a different starting player or an ordered variant cannot substitute for the game obligations used in the theorem.

5.2 Corruption tests and trusted components

For each original five-color certificate the public command makes seven semantic changes: it omits an opponent reply, reverses the declared winner, inserts an illegal Alice move, adds an edge, changes the starting player, treats the root as a terminal node, and adds a move at a terminal node. Each altered certificate is submitted directly to both checkers, and both must reject it. The command therefore requires 28 rejection outcomes in addition to accepting the two unchanged certificates. These controls exercise game semantics rather than file-integrity checks.

The six-color Alice certificate follows the opening in Lemma 3 and then selects the smallest uncolored label on Alice's turns. Every Bob reply is retained. All its leaves are full colorings, so no early threshold stop replaces unrestricted greedy completion. Both checkers reject eight changes: a missing Bob reply, an Alice pass, a nonleast greedy color, ordered play, Bob moving first, an unfinished leaf, threshold five, and a missing successor. The local-fact check also rejects a deleted clique edge, a violation of the degree bound, and a cyclic shift of the vertex labels.

The proof relies on the correctness of the two Python implementations, the Python interpreter and its JSON and gzip libraries, and the written finite-induction and blocked-vertex arguments

above. The certificate checks use exact integer and finite-set operations. The graph drawing is generated from the graph data, and its edges are checked against that data; the mathematical definition of G remains the edge list. The checks in this bundle run in Python. The search that found the graph is unnecessary for checking the theorem, and no exhaustive graph search or claim of minimality is part of the result.

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