A counterexample on two kinds of positive dependence

P. Graham
https://orcid.org/0009-0007-8493-1505

October 31, 2026

Abstract

We construct a strict bivariate Archimedean copula that is positively quadrant dependent but is not positive measure inducing. This disproves the forward implication in the equivalence conjectured by Sebastian Fuchs and Marco Tschimpke. Their work introduced the positive measure inducing property through a reflected copula sum and established the equivalence for the Frank family. Our example separates the two properties in the full Archimedean class. Its inverse generator is the maximum of two exponential functions. Convexity gives copula validity, and a piecewise linear inequality in logarithmic coordinates gives positive quadrant dependence throughout the unit square. A rectangle strictly inside the open square $(0, 1/2)^2$ has a negative increment for the reflected sum, excluding the positive measure required by the definition. All corner estimates use rational arithmetic and squaring, without a density assumption. The complete argument, including generator admissibility, boundary values and the measure-theoretic contradiction, is formalized in Lean. The converse implication and the separate negative-dependence equivalence are not resolved by this example.

This work has not yet been reviewed by experts. The proofs, the Lean formalization and the text were produced by AI models (from OpenAI and Anthropic), working under the author's direction.

Mathematics Subject Classification (2020): 60E05, 62H05.
Keywords: Archimedean copula, positive quadrant dependence, positive measure inducing dependence.

1 Introduction

Theorem 1. *For $t \geq 0$ and $0 < u \leq 1$, put*

$$\psi(t) = \max\{e^{-t}, \tfrac{1}{4}e^{-t/2}\}, \qquad \phi(u) = \max\{-\log u, -2\log u - \log 16\}.$$

The function ϕ, extended by $\phi(0) = \infty$, is a strict Archimedean generator in the sense of [1], and ψ is its inverse on $[0, \infty)$. The function on $[0, 1]^2$ defined by

$$C(u, v) = \begin{cases} \psi(\phi(u) + \phi(v)), & u, v > 0, \\ 0, & uv = 0 \end{cases}$$

is a copula, satisfies $C(u, v) \geq uv$ everywhere, and is not positive measure inducing in the sense of Fuchs and Tschimpke.

A copula is a two-increasing function on the unit square, zero on the lower edges and with uniform margins $C(u,1) = u$ and $C(1,v) = v$. The inequality in the theorem is positive quadrant dependence (PQD). For example, the independence copula $\Pi(u,v) = uv$ has PQD with equality. For any function F of two variables write its rectangle increment as

$$\Delta F[a,b;d,e] = F(b,e) - F(a,e) - F(b,d) + F(a,d).$$

Two-increasing means that these increments are nonnegative whenever $a \le b$ and $d \le e$ in the unit interval.

Fuchs and Tschimpke's Definition 1 and equation (2.2) in [1], with numbering from the arXiv version specified there, define the positive measure inducing property (PMI) by requiring that

$$E_C(u,v) = C(u,v) + C(1-u,v) + C(u,1-v) + C(1-u,1-v) - 1 \tag{1}$$

induce a positive measure on $(0,1/2)^2$. Thus every half-open rectangle $(a,b] \times (d,e]$ compactly contained in that square must have nonnegative E_C-increment. For independence, E_Π is identically zero. Example 6 of [1] gives the PQD–PMI equivalence for Frank copulas. Conjecture 1 proposes it for all Archimedean copulas; Theorem 1 disproves the implication PQD $\Rightarrow$ PMI. The conjecture's reverse implication and its separate NMI equivalence are outside this conclusion.

The generator description is classical: the bivariate case of McNeil and Nešlehová's Theorem 2.2 in [2] uses decreasing convex inverse generators. We verify those properties and the copula axioms directly. The distinction between a positive measure and a density is essential: Remark 1(2) of [1] gives a density criterion under absolute continuity, whereas our proof uses Definition 1 itself. Lean checks the full statement from these definitions, as described in Section 5. The superadditivity criterion behind our PQD argument appears in Müller and Scarsini's Theorem 2.7 in the working version of [3], attributed there to Joe, *Multivariate Models and Dependence Concepts* (1997), page 109. Fernández Sánchez and Trutschnig's Corollary 4 in the preprint of [4] expresses the mass of an Archimedean level curve in terms of one-sided generator derivatives. That formula motivates attention to corners; here direct rectangle increments suffice.

2 Proof overview

The maximum of two exponentials remains convex and strictly decreasing. In logarithmic coordinates its inverse has two affine branches, whose maximum is superadditive. These two observations establish the global copula and PQD assertions. The separation of PMI occurs at the change of branch: near a reflected corner, the copula is a maximum of a product and a square root of that product. The alternating sum of sixteen corner values over one small rectangle is negative. Exact rational enclosures make this last sign calculation elementary and avoid differentiating across the change of branch.

3 The generator and positive quadrant dependence

Lemma 2. *On $(0,1]$ and $[0,\infty)$, respectively, the functions ϕ and ψ in Theorem 1 are continuous, strictly decreasing and convex. They are inverse to each other, with $\phi(1) = 0$, $\phi(u) \to \infty$ as $u \downarrow 0$, $\psi(0) = 1$ and $\psi(t) \to 0$ as $t \to \infty$. Their construction defines a copula on the closed unit square.*

Proof. Each branch has the claimed continuity, monotonicity and convexity; taking a maximum preserves these properties, including strict decrease. The branches of ψ meet at $t = \log 16$, at value

$1/16$. Inverting on the two sides gives $-\log u$ for $u \geq 1/16$ and $-2\log u - \log 16$ for $u \leq 1/16$, exactly ϕ. The endpoint values and limits follow from these formulas.

The inverse identities give the margins, and the definition gives the zero edges. For a positive rectangle with $a \leq b$ and $d \leq e$ set

$$s = \phi(b) + \phi(e), \quad h = \phi(a) - \phi(b) \geq 0, \quad k = \phi(d) - \phi(e) \geq 0.$$

Its C-increment is $\psi(s+h+k) - \psi(s+h) - \psi(s+k) + \psi(s)$, which is nonnegative: for a convex function, an increment of fixed nonnegative length is nondecreasing in its starting point. If $a = 0$, the increment is $C(b,e) - C(b,d) \geq 0$ by monotonicity; the case $d = 0$ is the same, and repeated endpoints give zero. Thus all closed-square rectangle increments are nonnegative. The margins and groundedness also imply $0 \leq C(u,v) \leq \min(u,v)$. □

Lemma 3. *The copula in Theorem 1 satisfies $C(u,v) \geq uv$ for every $(u,v) \in [0,1]^2$.*

Proof. Let $L = \log 4$ and $f(x) = \max(x, 2x - 2L)$ for $x \geq 0$. This function is superadditive. If $x, y \leq 2L$, then $f(x) + f(y) = x + y \leq f(x+y)$. If $x \geq 2L \geq y$, then $f(x) + f(y) = 2x - 2L + y \leq 2(x+y) - 2L \leq f(x+y)$. The symmetric case is identical. If both exceed $2L$, the sum is $2(x+y) - 4L \leq f(x+y)$. For positive $u, v \leq 1$, take $x = -\log u$ and $y = -\log v$. Then $\phi(u) + \phi(v) \leq \phi(uv)$, so the decreasing inverse gives $C(u,v) \geq \psi(\phi(uv)) = uv$. The zero edges give equality. □

4 A negative reflected rectangle

Set

$$a = \frac{63}{256}, \quad b = \frac{65}{256}, \quad d = \frac{61}{768}, \quad e = \frac{67}{768}.$$

These endpoints satisfy $0 < a < b < 1/2$ and $0 < d < e < 1/2$. Every coordinate of each reflected corner lies in $[1/16, 1]$. On this interval $\phi(u) = -\log u$, hence

$$C(u,v) = \max\{uv, \sqrt{uv}/4\}. \tag{2}$$

Put $D = 196608$ and $q(n) = \sqrt{n/D}/4$. Table 1 lists all sixteen values needed in (1). The branch follows by comparing uv with $1/16$, or its numerator with $D/16 = 12288$.

(u,v)	$C(u,v)$	$C(1-u,v)$	$C(u,1-v)$	$C(1-u,1-v)$
(b,e)	$q(4355)$	$12797/D$	$45565/D$	$133891/D$
(a,e)	$q(4221)$	$12931/D$	$44163/D$	$135293/D$
(b,d)	$q(3965)$	$q(11651)$	$45955/D$	$135037/D$
(a,d)	$q(3843)$	$q(11773)$	$44541/D$	$136451/D$

Table 1: Reflected corner values for the negative rectangle.

Lemma 4. *For the displayed rectangle, $\Delta E_C[a,b;d,e] < 0$.*

Proof. The following integers k satisfy $k/10^6 \leq q(n) \leq (k+1)/10^6$:

n	4355	4221	3965	11651	3843	11773
k	37207	36630	35502	60858	34952	61176

Each enclosure follows by squaring nonnegative numbers: its two checks are $16Dk^2 \le 10^{12}n \le 16D(k+1)^2$. In the alternating sum of Table 1, use upper endpoints in the first and fourth rows, and lower endpoints in the second and third rows. The constant terms in (1) cancel, and this gives

$$\Delta E_C[a,b;d,e] \le \frac{37208 - 36630 - 35502 - 60858 + 34953 + 61177}{10^6} - \frac{134}{196608} = -\frac{512347}{1536000000} < 0.$$

□

Proof of Theorem 1. Lemma 2 establishes strict generator admissibility and all copula axioms, and Lemma 3 proves PQD. If C were PMI, its induced positive measure on the open square $(0,1/2)^2$ would assign the rectangle $(a,b]\times(d,e]$ the value $\Delta E_C[a,b;d,e]$. Lemma 4 makes this value negative, a contradiction. □

5 Formal verification

The complete theorem has the following Lean statement, in namespace `FT23`:

```
theorem conjecture_counterexample :
    StrictGenerator phi psi ∧ IsCopula C ∧
    PQD C ∧ ¬ PrimaryPMI C
```

Here the logical symbols are conjunction and negation. `StrictGenerator` records the continuity, strict decrease, convexity, inverse identities and endpoint limits of both displayed functions. `IsCopula` includes groundedness, uniform margins and two-increasingness for all rectangles in the closed square, including repeated endpoints. `PQD` quantifies over that whole square. `PrimaryPMI` asserts the existence of a positive Borel measure on the open square $(0,1/2)^2$ whose finite half-open rectangle values equal the corresponding increments of E_C. Thus its negation has the measure-theoretic meaning used above, rather than merely rejecting a smooth density test. Appendix A quotes the definitions and makes the endpoint convention explicit.

The formal proof establishes the generator identities and analytic inequalities, then bounds the same sixteen corner values by exact rational numbers. Its enclosures are tighter than the displayed ones. The independence example checks that E_Π and all its increments vanish. Only Lean's standard axioms of propositional extensionality, classical choice and quotient soundness occur. The build checks this axiom list and the final theorem's type; no compiled numerical evaluation is used.

The accompanying Lean project contains all proof sources and pinned library dependencies. Its README gives the standard Lake commands; after the dependencies are available, checking takes a few minutes. Reproducing the theorem requires Lean and mathlib but no search code, external certificate, or unproved analytic premise.

References

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[2] A. J. McNeil and J. Nešlehová, *Multivariate Archimedean copulas, d-monotone functions and ℓ_1-norm symmetric distributions*, Annals of Statistics **37** (2009), 3059–3097. doi:10.1214/07-AOS556. Theorem 2.2; arXiv:0908.3750v1.

[3] A. Müller and M. Scarsini, *Archimedean copulae and positive dependence*, Journal of Multivariate Analysis **93** (2005), 434–445. doi:10.1016/j.jmva.2004.04.003. Theorem 2.7 in the 2003 working version.

[4] J. Fernández Sánchez and W. Trutschnig, *Singularity aspects of Archimedean copulas*, Journal of Mathematical Analysis and Applications **432** (2015), 103–113. doi:10.1016/j.jmaa.2015.06.036. Corollary 4 in the June 22, 2015 preprint.

Appendix A Lean definitions

The following declarations are quoted verbatim from the formalization, in namespace `FT23`; `Set`, `Filter` and `MeasureTheory` are open. The real-valued Lean `phi` is used only on $(0,1]$: since `Real.log 0 = 0`, its value at zero is zero, while the extended value $\phi(0)=\infty$ is represented by the `StrictGenerator.phi_at_zero` right-limit condition; the zero edges of `C` are defined directly.

The definition of `C` is the strict Archimedean construction of [1, Section 2], with those zero edges.

```
def C (u v : ℝ) : ℝ := if 0 < u ∧ 0 < v then psi (phi u + phi v) else 0

def delta (F : ℝ → ℝ → ℝ) (a b d e : ℝ) : ℝ :=
  F b e - F a e - F b d + F a d
```

Here `delta` is the half-open rectangle increment used to interpret the induced measure in [1, Definition 1]. The reflection definitions below give `E`; the proved identity `E_expanded` is exactly [1, equation (2.2)].

```
def nu1 (F : ℝ → ℝ → ℝ) (u v : ℝ) : ℝ := v - F (1-u) v
def nu2 (F : ℝ → ℝ → ℝ) (u v : ℝ) : ℝ := u - F u (1-v)
def nu (F : ℝ → ℝ → ℝ) (u v : ℝ) : ℝ := u + v - 1 + F (1-u) (1-v)
def E (F : ℝ → ℝ → ℝ) (u v : ℝ) : ℝ :=
  F u v - nu1 F u v - nu2 F u v + nu F u v

lemma E_expanded (F : ℝ → ℝ → ℝ) (u v : ℝ) :
    E F u v = F u v + F (1-u) v + F u (1-v) + F (1-u) (1-v) - 1 := by
  unfold E nu1 nu2 nu
  ring
```

The definition of `PQD` is the inequality in [1, Section 2], extended to the closed square, where the margins and zero edges give the boundary cases.

```
def PQD (F : ℝ → ℝ → ℝ) : Prop :=
  ∀ u ∈ Icc (0:ℝ) 1, ∀ v ∈ Icc (0:ℝ) 1, u*v ≤ F u v
```

The predicate `ValidRect` places all four endpoints strictly inside the square of [1, Definition 1]; `HalfSquare` is that square as a subtype, and `rect` is a half-open rectangle.

```
def ValidRect (a b d e : ℝ) : Prop :=
  0 < a ∧ a < b ∧ b < 1/2 ∧ 0 < d ∧ d < e ∧ e < 1/2

def halfSquare : Set (ℝ × ℝ) := Ioo (0:ℝ) (1/2) ×ˢ Ioo (0:ℝ) (1/2)
abbrev HalfSquare := {z : ℝ × ℝ // z ∈ halfSquare}
def rect (a b d e : ℝ) : Set (ℝ × ℝ) := Ioc a b ×ˢ Ioc d e
```

The predicate `PrimaryPMI` is the necessary rectangle-value consequence of [1, Definition 1]: a positive Borel measure on the open square has each indicated finite value equal to the corresponding increment of `E`. The clause $\neq \top$ excludes infinite measure before `toReal` converts its value to a real number; negating this necessary condition therefore rules out PMI itself.

```
def PrimaryPMI (F : ℝ → ℝ → ℝ) : Prop :=
  ∃ μ : Measure HalfSquare, ∀ a b d e : ℝ, ValidRect a b d e →
    μ (Subtype.val ⁻¹' rect a b d e) ≠ ⊤ ∧
    (μ (Subtype.val ⁻¹' rect a b d e)).toReal = delta (E F) a b d e
```