Sharp linear bounds for the EKG sequence

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October 26, 2026

Abstract

We prove that the EKG sequence satisfies $13n/28 \le a(n) \le 12n/7$ for every $n \ge 1$, with equality only at $n = 28$ and $n = 7$, respectively. Discovered by Jonathan Ayres, the sequence starts with $1, 2$ and repeatedly takes the least unused positive integer sharing a factor greater than 1 with its predecessor. This proves the sharp linear bounds conjectured by Lagarias, Rains and Sloane, who established coarse bounds at every index. Hofman and Pilipczuk proved the limits $1/2$, $3/2$ and 1 of $a(n)/n$ on primes, their triples and the other terms, and Haver made these estimates explicit and formalized them in Lean. Our earlier second-order asymptotic also builds on Haver's work. We sharpen Haver's self-improving early-even bound with a shorter prime window and effective large-sieve constants, and control odd terms so that the analytic range meets a finite computation. Lean checks the deduction from the definition, conditional on computed premises; independent programs check those premises, and their correspondence with the formal statements was checked by reading the code.

This work has not yet been reviewed by experts. The proofs, the Lean formalization, the programs and the text were produced by AI models (from OpenAI and Anthropic), working under the author's direction.

2020 *Mathematics Subject Classification*: Primary 11B83; Secondary 11N36, 68V15.
Keywords: EKG sequence, greedy permutation, explicit bounds, large sieve, formal verification.

1 Introduction

The EKG sequence, discovered by Jonathan Ayres and studied by Lagarias, Rains and Sloane [12, Introduction], is A064413 in the OEIS [18]. Write $U_t = \{a(1), \dots, a(t)\}$. The sequence is defined by

$$a(1) = 1, \quad a(2) = 2, \quad a(n) = \min\{m \ge 1 : \ m \notin U_{n-1}, \ \gcd(m, a(n-1)) > 1\} \quad (n \ge 3), \qquad (1)$$

so it begins $1, 2, 4, 6, 3, 9, 12, 8, 10, 5, 15, 18, 14, 7, 21, 24, 16, 20, 22, 11, \dots$. It is a permutation of the positive integers [12, Theorem 1] (Theorem 2.5 in the journal version). Its graph has three visible lines (Figure 1): most terms satisfy $a(n) \approx n$, while each odd prime p is immediately preceded by $2p$ and followed by $3p$, and when $a(n) = p$ we have $p \approx n/2$ (and $a(n+1) = 3p \approx 3n/2$). Lagarias, Rains and Sloane proved $n/260 \le a(n) \le 14n$ and conjectured sharp linear bounds [12, Conjecture 3, end of §4] (Conjecture 4.3 in the journal version):

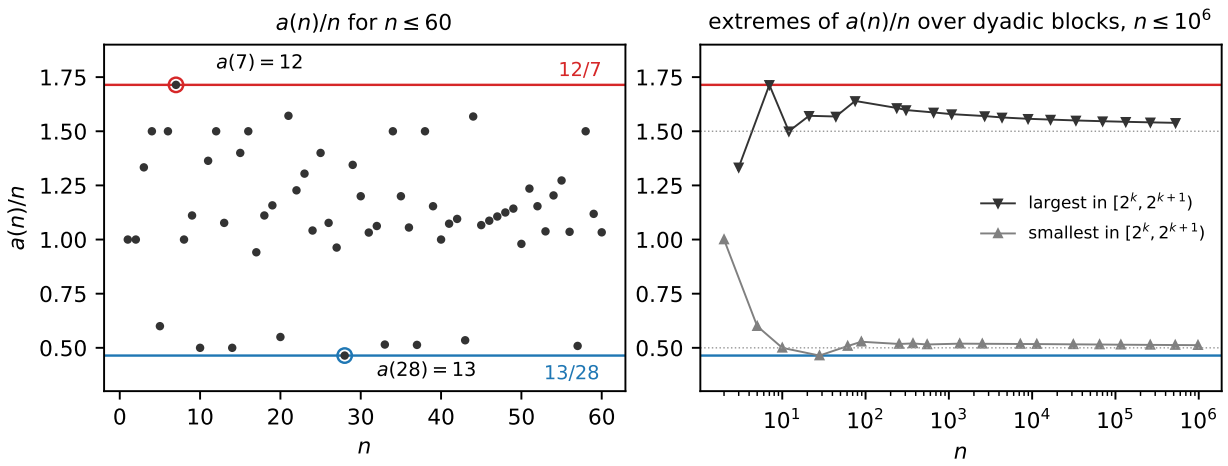


Figure 1: Left: $a(n)/n$ for $n \le 60$, with the bounds 12/7 (top) and 13/28 (bottom) of Theorem 1, attained at $n = 7$ and $n = 28$. Right: the largest and smallest values of $a(n)/n$ over each block $2^k \le n < 2^{k+1}$, $n \le 10^6$, approaching 3/2 and 1/2 (dotted). Drawn from the computed sequence.

The sequence $a(n)$ satisfies $a(n) \ge \frac{13}{28}n$, with equality if and only if $n = 28$, and $a(n) \le \frac{12}{7}n$, with equality if and only if $n = 7$.

Theorem 1. *For every integer $n \ge 1$,*

$$\frac{13}{28}\,n \;\le\; a(n) \;\le\; \frac{12}{7}\,n.$$

The lower bound is an equality only for $n = 28$ (where $a(28) = 13$) and the upper bound only for $n = 7$ (where $a(7) = 12$).

Prior work. Lagarias, Rains and Sloane give the pointer rule as an alternative definition [12, eq. (2)] and prove the coarse bounds above [12, Theorems 2–3]. Their Conjecture 2 states $a(n) \sim n(1 + 1/(3\log n))$ when $a(n)$ is neither p nor $3p$ for a prime p, $a(n) \sim (n/2)(1 + 1/(3\log n))$ when $a(n) = p$, and $a(n) \sim (3n/2)(1 + 1/(3\log n))$ when $a(n) = 3p$ [12, Conjecture 2]. Hofman and Pilipczuk [10, Theorems 4.3, 4.17, Lemma 4.5 and Corollary 4.12] proved the full pattern $2p, p, 3p$ for odd primes, bounded the crossings of a level by composite terms, and proved the leading limits $a(n)/n \to 1/2$, 3/2 and 1 for prime, triple-prime and remaining terms, respectively. Haver made these arguments explicit and formalized them in Lean [9, (F2), Theorems 5, 8, 11 and Corollary 12(b)], reducing the pattern's computed base to the primes $p \le 43$, bounding the number of used even numbers above the least missing even number, the spread of the terms above it, and the unused composites below it, and relating the index to that even number by a counting identity. The author [8] then obtained a second-order asymptotic, $n + n/(3\log n) + o(n/\log n)$, for a smoothed version of the sequence (each odd prime p and $3p$ replaced by $2p$), on Haver's Lean foundation. The present proof uses Haver's Lean definition of the sequence and Haver's estimates, and the author's exact form of the counting identity, as proved Lean theorems.

Haver already closes a self-improving early-even inequality and obtains its leading asymptotic constant by Brun–Titchmarsh [9, Lemma 4, Theorem 5, Remark 6(b)]. The hardest step is to make

that inequality close at a frontier reached by a finite computation; Haver's explicit bound is too weak at that frontier. We shorten the prime window from $4K+3$ to $3K+2$ integers, where K counts the used even numbers above the least missing even number; effective large-sieve constants and odd-term control make the analytic range meet a finite computation.

The proof uses Lean and computations of a finite prefix and an interval-arithmetic certificate. Section 9 gives the formal statement and its relation to the theorem; Appendix B describes the computed premises and their checks.

The result is computer-assisted.

2 Notation and known facts

All logarithms are natural. The notation below separates the three kinds of terms and the errors in the counting identity. The terms *frontier*, *early evens* and *tick value* follow Haver [9, §§1–2]. A *state* $t \geq 1$ is the moment after $a(1), \ldots, a(t)$ have been placed. For a state t define:

- $E(t)$, the *frontier*: the least positive even integer not in U_t. It is nondecreasing in t, and every even number below $E(t)$ is used.
- $K(t)$, the number of *early evens*: even $w \in U_t$ with $w > E(t)$.
- $B_q(t)$, the least multiple of q not in U_t.
- A *tick value* is $3p$ with p an odd prime. $T(t)$ is the largest element of U_t that is not a tick value, and $O(t)$ is the number of odd $w \in U_t$ with $w > E(t)$ that are not tick values.
- $H(t)$, the number of *holes*: composite $h < E(t)$ with $h \notin U_t$.
- $A(x) = \pi(x) - 2\pi(x/2) + \pi(x/3)$, where π counts primes.

For example, after $a(1..7) = 1, 2, 4, 6, 3, 9, 12$ we have $E(7) = 8$, the early even 12, so $K(7) = 1$, and $O(7) = 0$ since 9 is a tick value.

We use the following facts; each is a Lean theorem in the foundation (Section 9). The parenthesized labels refer to the statements recalled here.

(F1) *Pointer rule* [12, eq. (2)]. $a(n) = \min\{B_q(n-1) : q \text{ prime},\ q \mid a(n-1)\}$ for $n \geq 3$.

(F2) *Prime pattern.* For each odd prime p there is an index N_p with $a(N_p) = 2p = E(N_p - 1)$, $a(N_p + 1) = p$ and $a(N_p + 2) = 3p$; the value $3p$ occurs only at $N_p + 2$, and indices N_p of distinct primes differ by at least 4. See [10, Theorem 4.3] and [9, (F2)]; the finite checks and detailed attributions are in Appendix B.1.

(F3) *Crossings* [10, Lemma 4.5], [9]. For a level $x > 2$, at most $\pi(\sqrt{x})$ indices n have $a(n)$ composite and $a(n) < x \leq a(n+1)$.

(L7) [9, Lemma 7] $B_3(t) \leq E(t) + 6K(t) + 6$ for $t \geq 9$.

(SP) *Span* [9, Theorem 8]. With $k = K(t) + 1$ and $e = E(t)$, $T(t) - e \leq 2k^2 + 2k\sqrt{k^2 + e}$ for $t \geq 9$.

(EE) Our variant of [9, Lemma 4], from its proof. Call $i < t$ a *prime up* at state t if $a(i)$ is prime and $a(i) < E(t) \le a(i+1)$; it is *paired* if $a(i+1)$ is odd and $a(i+2) < E(t)$, and *unpaired* otherwise. Then $K(t) \le \pi(\sqrt{E(t)})+1+\#\{\text{unpaired prime ups at } t\}$. Pairing uses only terms already placed at state t, so $i+2 \le t$ is required; a prime up with $i+1 = t$ is unpaired.

(CL) *Counting identity* [9, Corollary 12(b)], in the exact form of [8]. For $t \ge 9$ and $e = E(t)$,

$$e - t = A(e) + H(t) - K(t) - O(t) + c(t), \qquad 1 \le c(t) \le 5.$$

Here $c(t)$ is the bounded correction in (CL). A crossing of a level is a step from one side of that level to the other; upward and downward refer to its direction. The inequalities in (F3) fix the convention at an endpoint.

Write $\pi^-(x) \le \pi(x) \le \pi^+(x)$ for the explicit prime-counting bounds in Appendix A; $\pi(x)$ counts primes up to x. The following two forms of the large sieve are proved in Lean.

(BT) *Primes in short intervals.* For real $x > 0$, $y \ge 1$ and integers $Q \ge 6$,

$$\pi(x+y) - \pi(x) \le \frac{y + Q^2}{\log(Q+1) + 1.07} + \pi(Q).$$

We use the additive inequality of Montgomery and Vaughan [16, Theorem 1] and their arithmetic lower bound [16, (2.7)], which they credit to Montgomery [15]. OpenAI's formalization [17] already supplies the arithmetic large sieve with Farey separation $1/Q^2$. For (BT) we use separation $1/(Q^2-1)$ to obtain the numerator $y + Q^2$. We also prove $G(Q) = \sum_{d \le Q} \mu^2(d)/\varphi(d) > \log(Q+1) + 1.07$, the form in the proof of [16, Lemma 7], whose printed statement is $G(z) \ge \log z + 1.07$ for real $z \ge 6$.

(LS) *Arithmetic large sieve* [15, 16]. For an integer $Q \ge 2$, a set of integers in an interval of N consecutive integers that avoids one residue class modulo every prime $p \le Q$ has at most $(N + Q^2)/\log Q$ elements.

3 Strategy

The three kinds of terms behave differently, and each is located through the frontier. If $a(n) = p$ is an odd prime then $n = N_p + 1$ and $E(n-2) = 2p$ by (F2); if $a(n) = 3p$ then $n = N_p + 2$ and $E(n-3) = 2p$; every other term either lies above the frontier $E(n-1)$, and then at most $T(n)$, or fills a hole below it, and holes lie close to the frontier. The counting identity (CL) relates the index t to $e = E(t)$: $A(e) \approx e/(3\log e)$ is small, and if K, O and H are small compared to e, then $t \approx e$ and the three kinds of terms give $a(n)/n$ near $1/2$, $3/2$ and 1.

Everything therefore rests on explicit upper bounds for K, O and H at every state, uniformly in a range of e. Haver bounds each by a multiple of $\pi(\sqrt{e})$ (times further factors), so the bounds are relatively small only when e is astronomically large. We split the indices into three ranges:

1. $n \le L = 1.6 \cdot 10^{10}$: computed (Section B.1);
2. $L < n < 10^{203}$: a bootstrap bound $K(t) \le M(E(t))$ for a step function M, certified with $E(L-2) \ge e_0 = 1.3328 \cdot 10^{10}$ and on every interval of a cover of $[e_0, 10^{205})$ (Sections 4–6);

3. $n \geq 10^{156}$: Haver's bounds alone (Section 7).

The ranges overlap, so every n is covered. Range 2 requires the bound on K already at $e = 1.3328 \cdot 10^{10}$.

The early-even bound. Theorem 1 needs sharp constants at every n: the lower line $a(n)/n \to 1/2$ sits only $1/28$ above $13/28$. With Haver's bound $K \leq 12\pi(\sqrt{E}) + 720$ [9, Theorem 5] the argument closes only for very large n; we use it that way for $n \geq 10^{156}$ (Lemma 8), far beyond any computation. The quantitative improvement uses a shorter prime window (Lemma 2, Corollary 3): the primes responsible for the excess of K lie among the $3K + 2$ integers just below $E/2$, so $K \leq \pi(\sqrt{E}) + 2 + \#\{\text{primes among } 3K + 2 \text{ consecutive integers}\}$, and a Brun–Titchmarsh bound from the large sieve turns this into $K \leq (1 + o(1))\pi(\sqrt{E})$. The bound is proved by induction over the moments of the sequence, so it needs no prior bound on K: at each moment the earlier ones already satisfy it, and an explicit inequality, checked on intervals of E, carries it to the next. At $E = 1.3328 \cdot 10^{10}$ the certified bound is 28,017, against 131,628 from Haver's bound. A second large-sieve bound controls the odd terms above E (Lemma 5). With these, explicit estimates close once $E \geq 1.3328 \cdot 10^{10}$; the computed first $1.6 \cdot 10^{10}$ terms reach that point (with E above $1.41 \cdot 10^{10}$ by $n = 1.393 \cdot 10^{10}$) and settle every smaller n.

4 Early evens

We bound the primes responsible for early evens. For a state t let $K^<(t) = \max\{K(s) : 1 \leq s < t\}$, with $K^<(1) = 0$.

Lemma 2. *Let t be a state and $e = E(t)$. Every unpaired prime up i at state t has $p = a(i)$ prime with $2p < e < 2p + 6K^<(t) + 6$.*

Proof. The prime 2 is never a prime up: $a(2) = 2$, $a(3) = 4$, and $E(t) \geq 6$ for $t \geq 3$. So p is odd, and by (F2) $a(i-1) = 2p = E(i-2)$ and $a(i+1) = 3p$. Since $2p \in U_t$, $2p \neq e$, and E is nondecreasing, so $2p < e$. Unpairedness forces every multiple of 3 below e to be used by the state $k = i + 1$ when $k < t$ (otherwise $a(i+2) \leq B_3(k) < e$ by (F1), since $3 \mid a(k)$); in particular $3 \nmid e$, as e is still unused at t. Each multiple w of 6 with $2p < w < e$ is then used by state k, is none of $2p$, p, $3p$, and so was already used at state $i - 2$, where $E(i-2) = 2p < w$: it is an early even at state $i - 2 < t$. Since $6 \nmid e$ there are more than $(e - 2p)/6 - 1$ such w, so $(e - 2p)/6 - 1 < K(i-2) \leq K^<(t)$. In the remaining case $i + 1 = t$ we have $U_t = U_{i-2} \cup \{2p, p, 3p\}$, so every even w with $2p < w < e$ lies in U_{i-2} and is an early even at state $i - 2$; hence $(e - 2p)/2 - 1 \leq K^<(t)$, which is stronger. □

Corollary 3 (Bootstrap)**.** *Let $e_0 < E_1$, let M be a nondecreasing function on the integers, and let $\mathrm{BT}(y)$ bound the number of primes in every block of y consecutive integers. Suppose that $K(s) \leq M(E(s))$ at every state s with $E(s) < e_0$, and that*

$$\pi(\sqrt{e}) + 2 + \mathrm{BT}(3M(e) + 2) \leq M(e) \tag{2}$$

for every even e with $e_0 \leq e < E_1$. Then $K(t) \leq M(E(t))$ at every state t with $E(t) < E_1$.

Proof. Strong induction on t. The states $t \leq 9$ have $K(t) \leq 1$ by direct computation of the first terms, and (2) gives $M(e) \geq 2$, so they satisfy the claim. Let $e = E(t) \geq e_0$. Every $s < t$

has $E(s) \le e$, so by induction and monotonicity $K^<(t) \le M(e)$. By Lemma 2 the primes p of the unpaired prime ups at t satisfy $e/2 - 3M(e) - 3 < p < e/2$, so they lie among $3M(e)+2$ consecutive integers, and they are distinct because the sequence does not repeat values. Thus (EE) gives $K(t) \le \pi(\sqrt{e}) + 1 + \mathrm{BT}(3M(e)+2) \le M(e)$ by (2). □

The function $\mathrm{BT}(y)$ bounds primes in an interval of length y; the label (BT) refers to the displayed sieve estimate. With (BT), $\mathrm{BT}(y) \le (y+Q^2)/(\log(Q+1)+1.07) + \pi(Q)$ for any integer $Q \ge 6$. For $y = 3M+2$ and Q near $\sqrt{y/\log y}$ this is about $2y/\log y \approx 6M/\log M$, so (2) holds once M is a little larger than $\pi(\sqrt{e})/(1-6/\log M)$. This is $(1+o(1))\pi(\sqrt{e})$ as $e \to \infty$, and about $2.6\,\pi(\sqrt{e})$ at $e = 1.3328 \cdot 10^{10}$. Haver's argument counted the same primes through their residues modulo 210, which bounds them only by a fixed fraction of the window; this is where Haver's factor 12 comes from.

5 Holes and odd terms above the frontier

The next two lemmas bound H and O in terms of a bound M on K. Lemma 4 is Haver's hole bound [9, Theorem 11] with an arbitrary nondecreasing bound M in place of Haver's explicit one, and with the hypothesis needed only at earlier states.

Lemma 4 (Holes)**.** *Let M be nondecreasing with $K(s) \le M(E(s))$ for every state $s < t$, and $e = E(t)$. Every hole $h \ge 15$ at state t satisfies $e - h \le 2\,\mathrm{spf}(h)\big(\pi(\sqrt{h}) + M(h) + 4\big)$, where $\mathrm{spf}(h)$ is the least prime factor of h, and $H(t) \le \pi(\sqrt{e})\big(\pi(\sqrt{e}) + M(e) + 5\big)$.*

Proof. A hole h is odd; put $s = \mathrm{spf}(h)$ and let t_1 be the first state with $E(t_1) > h$. Every multiple w of $2s$ in (h, e) is used. If it was placed after t_1 then the next term is at most $B_s < h$ by (F1), a down-crossing of the level h; at most $K(t_1 - 1) \le M(h)$ of them were placed before t_1. Up-crossings come from composite bases (at most $\pi(\sqrt{h})$ by (F3)) or from primes p with $a(n) = 3p$, each of which is matched with its own down-crossing $2p \to p$. As ups and downs alternate, the number of multiples of $2s$ in (h, e) is at most $\pi(\sqrt{h}) + M(h) + 3$, which gives the first bound. Summing over the primes $s \le \sqrt{e}$ the number of odd multiples of s in the resulting window gives the second. □

Lemma 5 (Odd terms above the frontier)**.** *Let $t \ge 10$ be a state, $e = E(t)$, $K' = \max_{s \le t} K(s)$, and let $Y \ge 6K' + 7$ be an integer. Put $X = e + Y + \frac{1}{2}$ and $R = (Y + \frac{1}{2})/(2(K(t)+1))$. Then*

$$O(t) \le \left\lceil \frac{Y}{2} \right\rceil + \pi(\sqrt{X}) + Z,$$

where Z is the number of integers m with $X < m \le T(t)$ all of whose prime factors exceed R.

Proof. At most $\lceil Y/2 \rceil$ odd values lie in (e, X). Let $v = a(i) > X$ be another value counted by $O(t)$, and $c = a(i-1)$ its predecessor. If c were even, (F1) with $q = 2$ would give $v \le E(i-1) \le e$; if c were 1 or an odd prime, v would be 2 or a tick value (F2); if c were a tick value, (F1) with $q = 3$ and (L7) would give $v \le e + 6K' + 6 < X$. So c is an odd composite, not a tick value, with $c \le T(t)$. For each prime $r \mid c$, $B_r(i-1) \ge v$ by (F1), so the multiples of $2r$ in (e, v) are all early evens at t: there are at least $(v-e)/(2r) - 1$ of them, whence $r > R$. If $c < X$ the step $c \to v$ crosses the level X upward, which happens at most $\pi(\sqrt{X})$ times by (F3); otherwise c is counted by Z, and $v \mapsto c$ is injective. □

The proof uses (F3) at the real level X. For the cover we weaken its crossing term to $\pi(\sqrt{e+Y+1})$, as in the formal lemma, which counts the odd non-tick excess above the integer $e+Y$. The remaining odd values contribute at most $\lceil Y/2 \rceil$.

By (SP) the numbers counted by Z lie in an interval of at most $S-Y$ integers, $S = 2k^2 + 2k\sqrt{k^2+e}$, and they avoid the residue 0 modulo every prime $p \le R$; so (LS) gives $Z \le (S - Y + Q^2)/\log Q$ for every integer $3 \le Q \le R$.

6 The middle range

We now combine the lemmas using a finite table of parameters on adjacent intervals of values of $E(t)$. The inequalities in each row bound the terms of the counting identity. Put $e_0 = 13{,}328{,}000{,}000$, $K_{\mathrm{fin}} = 329$, $E_1 = 10^{205}$ and $L = 16{,}000{,}000{,}000$.

Definition 6. A *row* is a tuple of nonnegative integers $(e_a, e_b, M, Y, Q, h_c, b_q)$. With $k = M+1$, $S = 2k^2 + 2k\sqrt{k^2+e_b}$, $p_U = \pi^+(\sqrt{e_b})$,

$$\begin{aligned} O^* &= \lfloor (Y+1)/2 \rfloor + \pi^+(\sqrt{e_b + Y + 1}) + (S - Y + Q^2)/\log Q, \\ H^* &= p_U(p_U + M + 5), \\ A_{\min} &= \pi^-(e_a) - 2\pi^+(e_b/2) + \pi^-(e_a/3), \\ A_{\max} &= \pi^+(e_b) - 2\pi^-(e_a/2) + \pi^+(e_b/3), \end{aligned}$$

the row is *valid* if $e_a < e_b$, $6M + 7 \le Y < S$, $3 \le Q \le (Y + \frac{1}{2})/(2(M+1))$, $h_c \ge 15$, $b_q \ge 6$ and

$$\begin{aligned} &\text{(B)} && p_U + 2 + \frac{3M + 2 + b_q^2}{\log(b_q+1) + 1.07} + \pi^+(b_q) \le M, \\ &\text{(Hc)} && h_c + 2\sqrt{h_c}(\pi^+(\sqrt{h_c}) + M + 4) < e_a, \\ &\text{(P)} && M + O^* + 1 - A_{\min} < e_a/13, \\ &\text{(Tk)} && A_{\max} + H^* + 2 < e_a/8, \\ &\text{(Od)} && 7S + 12(A_{\max} + H^* + 4) < 5e_a, \\ &\text{(Ol)} && 13(e_b - A_{\min} + M + O^*) < 28h_c. \end{aligned}$$

A *cover* is a list of valid rows with e_a of the first row equal to e_0, $M \ge K_{\mathrm{fin}}$ in the first row, each e_a equal to the previous e_b, M nondecreasing, and last $e_b \ge E_1$.

In words: (B) is the bootstrap inequality (2) on the row; (Hc) puts every hole above h_c; (P) and (Tk) bound the index t above and below in terms of $E(t)$, which handles primes and tick values; (Od) bounds the terms above the frontier, and (Ol) the holes filled, for the remaining terms. O^*, H^*, $A_{\min}$ and $A_{\max}$ bound O, H and A on the row.

The accompanying interval certificate gives a cover with first row $M = 28{,}017$. Appendix B.3 displays its first row and describes the finite checks. The condition (P) in Definition 6 is an index inequality; Appendix A uses the same label for the prime-counting bounds.

Theorem 7. *Assume a cover exists, the bounds of Theorem 1 for $n \le L$ with its equality cases, $E(L-2) \ge e_0$, and $K(s) \le K_{\mathrm{fin}}$ at every state s with $E(s) < e_0$. Then every n with $L < n < 10^{203}$ has $13n < 28a(n)$ and $7a(n) < 12n$. Moreover every state t with $e_0 \le E(t) < E_1$ has $t + 2 < 14E(t)/13$.*

Proof. Let $M(x) = K_{\mathrm{fin}}$ for $x < e_0$ and $M(x) = M_j$ on the j-th row's interval $[e_a, e_b)$; it is nondecreasing. Condition (B) and (BT) with $Q = b_q$ give (2) on every row, so Corollary 3 yields $K(t) \le M(E(t))$ at every state with $E(t) < E_1$.

Fix a state t with $e = E(t)$ in the interval of row j. Then $K(s) \le M_j$ for all $s \le t$. Lemma 4 gives $H(t) \le H^*$, and every hole h exceeds h_c, because a hole $h \le h_c$ would give $e \le h_c + 2\sqrt{h_c}(\pi(\sqrt{h_c}) + M_j + 4) < e_a \le e$ by (Hc) (using $\mathrm{spf}(h) \le \sqrt{h}$). (SP) gives $T(t) - e \le S$, and Lemma 5 with Y and the sieve bound after it give $O(t) \le O^*$. Since $A_{\min} \le A(e) \le A_{\max}$, the identity (CL) yields

$$e - A_{\max} - H^* - 5 \;\le\; t \;\le\; e - A_{\min} + M + O^* - 1. \tag{3}$$

By (P), $t + 2 < e + e_a/13 \le 14e/13$; by (Tk), $t + 3 > 7e/8$.

Now let $L < n < 10^{203}$. Then $E(n) \le 2n + 2 < E_1$, and every state $t \ge n - 3$ has $E(t) \ge E(L - 2) \ge e_0$, so the bounds above apply at $t \in [n - 3, n]$.

- If $a(n) = p$ is an odd prime, take $t = n - 2$, so $e = 2p$. Then $13n = 13(t + 2) < 14e = 28p$, and $n > 7e/8 - 1$ gives $12n > 7p$.
- If $a(n) = 3p$, take $t = n-3$, so $e = 2p$. Then $12n > 12 \cdot 7e/8 = 21p$, and $13n < 14e+13 < 28 \cdot 3p$.
- Otherwise $v = a(n)$ is even or an odd composite that is not a tick value; take $t = n - 1$. If $v > e$ then $v \le T(n) \le e + S$, and (3) with (Od) gives $7v < 12n$, while $13n < 14e < 28v$. If $v = e$ both bounds follow from $7e/8 < n + 2$ and $13n < 14e$. If $v < e$, then v is a hole at state t, so $v > h_c$ and (Ol) with (3) gives $28v > 13n$, while $7v < 7e < 12n$. □

7 The far range

For very large n Haver's explicit bounds suffice, without a cover. Ordinary terms here mean even terms and odd composite terms other than the values $3p$. The hypothesis below is used only while $E(t) < E_1$, to obtain the stated lower bound on $E(t)$ at large indices. Once $E(t) \ge E_1$ that lower bound is automatic. The subsequent estimates use Haver's bounds; this is the sense of range 3 in the strategy. In the proof below, (P) means the prime-counting bounds of Appendix A. The numerical estimates follow by substituting those bounds into Haver's sequence estimates; the accompanying Lean proof contains their derivation.

Lemma 8. *Suppose every state t with $e_0 \le E(t) < E_1$ has $t + 2 < 14E(t)/13$, and $E(L - 2) \ge e_0$. Then every $n \ge 10^{156}$ has $13n < 28a(n)$ and $7a(n) < 12n$.*

Proof. At the states $t \ge n - 3$ the hypothesis forces $E(t) > 9 \cdot 10^{155}$. From (P) one derives $0.921\, x/\log x \le \pi(x) \le 1.045\, x/\log x$ for $x \ge 10^{76}$. Inserted into Haver's bounds $K \le 12\pi(\sqrt{e}) + 720$ [9, Theorem 5], (SP), Haver's hole bound and (CL) (with $O \le (T - e)/2 + 1$), this gives $0.997\, e < t$, $t + 3 < 1.0766\, e$ and $T(t) < 1.153\, e$ at each such state t, with $e = E(t)$, and shows that every hole exceeds $e/1.1543$. Then the three cases of the proof of Theorem 7 give $a(n)/n$ between $1/(1.1543 \cdot 1.0766) > 0.8$ and $1.153/0.997 < 12/7$ for ordinary terms, and between $1/(2 \cdot 1.0766)$ and $3/(2 \cdot 0.997)$ for primes and tick values; the binding comparison is $13 \cdot 2 \cdot 1.0766 < 28$. □

8 Proof of the main theorem

Proof of Theorem 1. The computed range supplies the three finite premises of Theorem 7, and the interval certificate supplies its cover. That theorem proves the strict bounds for $L < n < 10^{203}$ and the inequality on states required by Lemma 8. The lemma proves the strict bounds for $n \geq 10^{156}$. These ranges overlap and cover every $n > L$; the computed bounds for $n \leq L$ supply both equality cases. □

9 Formal verification

Lean 4 [7, 14] checks the theorem

$$\texttt{EKG3.conj3_of_computations} :$$
$$\texttt{FiniteFa} \to \texttt{FiniteFbP3} \to \texttt{FiniteFcP3} \to \texttt{CertifiedCoverP3} \to \texttt{Conj3Statement}.$$

Its conclusion says: for every $i \geq 0$, with $n = i + 1$ and $a(n) = \texttt{EKG.actualA}\ i$, $13n \leq 28a(n)$, $7a(n) \leq 12n$, $13n = 28a(n) \iff n = 28$ and $7a(n) = 12n \iff n = 7$. These integer forms are equivalent to the rational ones in Theorem 1. The sequence in the formal statement is Haver's greedy sequence with initial terms $1, 2$, injectivity, a shared prime factor and minimality as in (1); its indexing starts at zero. Reading the implication from left to right, the four premises are respectively (Fa), (Fb), (Fc) of Appendix B.1, and the cover of Definition 6. They assert the finite bounds, the lower bound on $E(L-2)$, the bound on K before e_0, and the existence of that cover, all about the sequence with no free parameters. The last term, `Conj3Statement`, is the conclusion just spelled out. Lean checks the analytic inputs and the deduction from these premises; the premises themselves and the correspondence of the programs with those definitions are checked outside Lean. The formal proof uses only Lean's standard axioms. Appendix B describes the computational checks and dependencies. Reproduction commands and implementation details are in the supplementary checking notes.

Related work. Haver's subsequent work on the Yellowstone permutation [1, 2, 3, 4, 5] concerns a different greedy permutation; nothing from those papers is used here. Other nearby visiting rules include Applegate, Havermann, Selcoe, Shevelev, Sloane and Zumkeller's Yellowstone permutation [20], Basistha and Ionaşcu's coprime permutation [21], and Na's real-multiplier process [22]. Nothing from these works is used here.

Appendix A Prime-counting bounds

These formulas specify the prime-counting bounds used in the interval conditions and in the far range. The analytic inputs are explicit and all are proved in Lean. Put $c = 1.03883$, $\alpha = 0.92129$ and, for $x \geq 256$, $F(x) = \frac{x}{\log x}\big(1 + \frac{1}{\log x} + \frac{2}{\log^2 x} + \frac{24}{\log^3 x}\big)$, $R(x) = c\sqrt{x} + \frac{c}{\log 2}x^{1/3}\log x$.

(P) *Prime counting.* For $x \geq 256$, $\pi^-(x) \leq \pi(x) \leq \pi^+(x)$, where $\pi^+(x) = cF(x) + J$ with $J = \pi(256) - \theta(256)/\log 256 + c\,(256/\log 256 - F(256))$, where $\pi(256) = 54$ and $\theta(256) = \sum_{p<256}\log p$, and $\pi^-(x) = \max\big(0, (\alpha x - 5\log x + 5 - R(x))/\log x\big)$; below 256 both equal $\pi(x)$. They follow by partial summation from Rosser and Schoenfeld's $\psi(x) < cx$ [19, Theorem 12]

and $\psi(x) \geq \alpha x - 5\log x + 5$ for $x \geq 30$, a weaker form of Chebyshev's $\psi(x) > Ax - \frac{5}{2}\log x - 1$, $A = 0.92129\ldots$ [6, §V], as proved in Lean in [11].

Appendix B Computational and formal details

B.1 Finite computation

The prime pattern. That p follows the first multiple of p is [12, Lemma 1] (Lemma 2.1 in the journal version), and that this multiple is $2p$ is [10, Theorem 4.3]. Hofman and Pilipczuk prove the latter by a crossing count at the level $2p$ that closes for $p > 25000$, and cover smaller p by the computation of 10^7 terms in [12]; Haver [9, Remark 3] closes the same count for every $p > 43$, leaving $p \leq 43$, which the first 82 terms show. The rest, $2p = E(N_p - 1)$, the uniqueness of $3p$ and the spacing, is proved in [9, (F2)]. In the Lean proof the primes $p \leq 43$ are checked on the first terms, $43 < p < 4096$ by a finite computation, and larger p by explicit prime bounds.

The premises of Theorem 7 about the first L terms are:

(Fa) for $1 \leq n \leq L$, $13n \leq 28a(n)$ and $7a(n) \leq 12n$, with equality only at $n = 28$ and $n = 7$;

(Fb) $E(L-2) \geq e_0$;

(Fc) $K(s) \leq 329$ at every state s with $E(s) < e_0$.

They are established by computing $a(1), \ldots, a(L)$ with the pointer rule (F1), keeping one pointer per prime, and tracking E and K. We find $E(L) = 16{,}230{,}216{,}746$. The frontier first reaches $1.413 \cdot 10^{10}$ at $n = 13{,}928{,}388{,}828 < L - 2$, which gives (Fb) since E is nondecreasing. The largest K at states with $E < 1.413 \cdot 10^{10}$ is 329; since every state with $E(s) < e_0$ comes before that crossing, this gives (Fc) for all states, not only those up to L. The only equality cases are $a(7) = 12$ and $a(28) = 13$.

Two programs written independently, with different factorization methods, memory layouts and formulas for K, compute these facts and agree; a third program checks the defining minimality property directly on the first 10^8 terms.

B.2 Formal development

The foundation is Haver's Lean archive [9] as extended in [8]. The analytic inputs reuse published formalizations: Rosser and Schoenfeld's Theorem 12 and a form of Chebyshev's lower bound from the IEANTN project [11] (with interval tools from LeanCert [13]), and the arithmetic large sieve from OpenAI's formalizations [17], specifically `OAI.Ostmann.publishedAdditiveLargeSieve` in `lean/OAI/NumberTheory/Ostmann/Preliminaries/AdditiveSieveProof.lean`. The port preserves the retained statements and proofs; its imports are adapted to the pinned local toolchain. OpenAI's arithmetic sieve already uses Farey separation $1/Q^2$. Building on that sieve, we use the stronger separation $1/(Q^2 - 1)$ for (BT), prove the bound $G(Q) > \log(Q+1) + 1.07$ from the proof of [16, Lemma 7], and derive (BT) and the bounds (P).

B.3 Interval certificate

The certificate has 16,857 rows. Its first row is shown in Table 1; the complete table accompanies the paper. Each row supplies the integer parameters of Definition 6. The independent checker

verifies the row inequalities, the common endpoints of consecutive intervals, monotonicity of M, and the first and last endpoints. The checker uses exact integers and rationals with outward-rounded interval arithmetic.

Parameter	First row
e_a	13,328,000,000
e_b	13,341,330,000
M	28,017
Y	266,194,600
Q	4,750
h_c	7,132,721,000
b_q	90

Table 1: The first row of the interval certificate.

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