Strongly cospectral sets in seven-dimensional cubelike graphs

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Abstract

Every set of pairwise strongly cospectral vertices in a cubelike graph on 128 vertices has at most four elements, and a connected graph attains four. Vertices are strongly cospectral if their orthogonal projections onto each adjacency eigenspace agree up to sign. This gives a negative answer to Árnadóttir and Godsil's question whether their upper bound is tight in every dimension: in dimension seven their bound gives eight. They proved the general bound, attained it in dimensions three through six, and supplied the connected construction attaining four used here. Our upper bound combines their abelian character criterion, building on Sin and Sorci's normal-Cayley criterion, with congruences, a second-moment bound and an exact finite inequality certified by linear programming duality. The universal upper bound, including the certificate checks, is formalized in Lean; exact computations also verify the attaining graph. Sin's construction gives arbitrarily large strongly cospectral classes as the dimension grows.

1 Introduction

A *cubelike graph* is a Cayley graph $\mathrm{Cay}(\mathbb{F}_2^d, S)$, where $S \subseteq \mathbb{F}_2^d \setminus \{0\}$: vertices x and y are adjacent when $x + y \in S$. For its adjacency matrix A, let E_θ be the orthogonal projection onto the eigenspace with eigenvalue θ, and let e_x be the coordinate vector at x. Vertices x, y are *strongly cospectral* when $E_\theta e_x = \pm E_\theta e_y$ for every eigenvalue θ, with the sign allowed to depend on θ.

Theorem 1.1. *Let $S \subseteq \mathbb{F}_2^7 \setminus \{0\}$. Every set of pairwise strongly cospectral vertices in $\mathrm{Cay}(\mathbb{F}_2^7, S)$ has at most four elements. This bound is attained by a connected graph.*

No connectivity or degree assumption is imposed on the upper bound; in particular, it includes the empty connection set.

Árnadóttir and Godsil [1, Corollary 8.3] proved that a pairwise strongly cospectral set in a cubelike graph on 2^d vertices has size at most $2^{\lceil d/2 \rceil - 1}$ for $d \geq 3$. Their Example 8.4 [1] attains this bound for $d = 3, 4, 5, 6$. In Section 11 [1] they asked:

> "Is the upper bound on the size of strongly cospectral sets in cubelike graphs tight in all dimensions?"

At $d = 7$ their bound is eight. Theorem 1.1 reduces it to four, and hence answers their question negatively.

The contribution here is the upper bound four in dimension seven. The character criterion used below is the specialization of Árnadóttir–Godsil's Theorem 10.1 to an elementary abelian 2-group, building on the normal-Cayley character criterion of Peter Sin and Julien Sorci [3, Theorems 2.3 and 2.4], as credited in Árnadóttir–Godsil's Theorem 5.2. The attaining connected example is Árnadóttir–Godsil's Theorem 10.2. Their bound follows from eigenvalue multiplicities and a global

second-moment argument [1, Theorems 6.1 and 8.2]. Here the hardest step is a stronger finite inequality, certified by linear programming duality. It combines disjointness of the eigenvalue values on eight character cosets with their individual means, their residues modulo four, and Parseval's identity. The earlier multiplicity bound does not impose these coset restrictions together.

Peter Sin [2, Theorem 4.4], in work first posted in 2022, constructed cubelike graphs with arbitrarily large strongly cospectral classes. The maximum in dimensions eight and higher is not determined here. The computational examples and Sage code of Árnadóttir and Godsil, and Gordon Royle's cubelike graph lists credited by them, are prior work [1, Appendices A–B and Acknowledgments].

The upper bound is proved in Lean, including the exact finite inequality; Section 8 describes its relation to the written proof.

Computer-assisted; not yet reviewed by experts. The proofs, the Lean formalization, the certificate checkers and the text were produced by AI models (from OpenAI and Anthropic), working under the author's direction.

2 Proof overview

The proof rules out a strongly cospectral class of size eight. Write $V = \mathbb{F}_2^7$. For a character indexed by $a \in V$, the corresponding adjacency eigenvalue is

$$\lambda_a = \sum_{s \in S} (-1)^{a \cdot s},$$

where the dot product takes values in $\mathbb{F}_2$. The vertices strongly cospectral with zero form a subspace $K(S)$ of V: a vector g lies in it precisely when $a \cdot g$ is constant on each *eigenvalue fiber*, meaning a set $\{a \in V : \lambda_a = \theta\}$. The sum $\sigma = \sum_{s \in S} s$ also belongs to $K(S)$. Consequently, if $|K(S)| \geq 8$, a change of basis places a three-dimensional subspace H inside $K(S)$ with $\sigma \in H$.

The eight cosets of the orthogonal complement $H^\perp$ then have disjoint sets of eigenvalue values. All values in a given coset have a prescribed residue modulo four. Their full coset means depend only on $B = S \cap H$, which has at most seven elements. After removing the trivial character from the zero coset, its remaining mean also depends on $n = |S|$. The sum of all squared eigenvalues is fixed by Parseval's identity: it is $128|S|$.

For each coset, retain only the nearest eigenvalue values on either side of its mean. These two endpoints give a lower bound for that coset's sum of squares. They must satisfy the residue restriction and must avoid the endpoints chosen by every other coset. The eigenvalue $|S|$ at the trivial character is treated separately. This leaves a finite question about eight pairs of integers, parameterized by $|S|$, B, and σ.

The hard finite inequality is that every permitted selection of these pairs has total lower bound strictly greater than $128|S|$. It is certified by assigning nonnegative penalties to endpoint values. The penalty-adjusted minimum for each coset can be computed independently; subtracting the total available penalty gives a valid lower bound because each endpoint value can be used by at most one coset. Exact certificates make that lower bound exceed $128|S|$ in every case. The resulting contradiction proves the upper bound.

3 The character criterion and normalization

We identify strongly cospectral classes with a subspace defined by character eigenvalues and put a hypothetical large class in a fixed position. Throughout the proof, $V = \mathbb{F}_2^7$, $S \subseteq V \setminus \{0\}$, and

$n = |S|$. For $a \in V$, define

$$\chi_a(x) = (-1)^{a \cdot x}, \qquad \lambda_a = \sum_{s \in S} \chi_a(s), \qquad n_a = |\{s \in S : a \cdot s = 1\}|. \tag{1}$$

In particular, $\lambda_a = n - 2n_a$. Define

$$K(S) = \{g \in V : \lambda_a = \lambda_b \implies a \cdot g = b \cdot g \text{ for all } a, b \in V\}. \tag{2}$$

Lemma 3.1 (Character criterion)**.** *For $x, y \in V$, the vertices x and y are strongly cospectral in* $\mathrm{Cay}(V, S)$ *if and only if $x + y \in K(S)$. The set $K(S)$ is a subspace, and its cosets are the strongly cospectral classes.*

Proof. The characters χ_a form an orthogonal basis of $\mathbb{R}^V$, with squared norm 128, and $A\chi_a = \lambda_a \chi_a$. Thus

$$E_\theta e_g = \frac{1}{128} \sum_{a : \lambda_a = \theta} \chi_a(g) \chi_a.$$

By linear independence, this equals $\pm E_\theta e_0$ precisely when $\chi_a(g)$ is constant over a with $\lambda_a = \theta$. This is exactly the condition $g \in K(S)$. Translations are graph automorphisms, so the assertion for x, y follows by translating x to zero. Finally, each condition defining $K(S)$ is a homogeneous linear equation in g, and hence $K(S)$ is a subspace. □

Lemma 3.2. *Put $\sigma = \sum_{s \in S} s$, where the sum is in V. Then $\sigma \in K(S)$, and for every $a \in V$,*

$$n_a \equiv a \cdot \sigma \pmod 2, \qquad \lambda_a \equiv n - 2(a \cdot \sigma) \pmod 4. \tag{3}$$

Here elements of $\mathbb{F}_2$ in integer congruences are represented by $0, 1$.

Proof. The parity of n_a is $\sum_{s \in S} a \cdot s = a \cdot \sigma$. Equality of two eigenvalues implies equality of the corresponding n_a, so it implies equality of the dot products with σ. The second congruence follows from $\lambda_a = n - 2n_a$. □

Let $e_1, \ldots, e_7$ denote the standard basis of V, and put $H = \langle e_1, e_2, e_3 \rangle$. We encode a vector $\sum_i x_i e_i$ by the integer $\sum_i 2^{i-1} x_i$. In particular, $H = \{0, \ldots, 7\}$. Addition of encoded vectors is bitwise exclusive-or, rather than integer addition.

Lemma 3.3 (Normalization)**.** *If $|K(S)| \geq 8$, an invertible linear change of coordinates replaces S by a connection set for which $H \subseteq K(S)$ and $\sigma \in H$.*

Proof. By Lemmas 3.1 and 3.2, $K(S)$ is a subspace of dimension at least three containing σ. Choose a three-dimensional subspace of $K(S)$ containing σ; this is possible also when $\sigma = 0$. Extend a basis of that subspace to a basis of V, and let T map this basis to the standard one. For the connection set TS,

$$\lambda^{TS}_a = \lambda^S_{T^\mathsf{T} a}, \qquad K(TS) = TK(S), \qquad \sigma(TS) = T\sigma(S).$$

The eigenvalue identity follows from $a \cdot Ts = (T^\mathsf{T} a) \cdot s$; substitution in (2) gives the identity for K. These identities give the required position. □

If a pairwise strongly cospectral set has more than four vertices, Lemma 3.1 gives $|K(S)| > 4$. The cardinality of a binary subspace is a power of two, so it is at least eight. It therefore suffices to exclude the normalized situation of Lemma 3.3.

4 Coset means, residues, and the second moment

We now derive the restrictions that cannot hold together in dimension seven. Assume $H \subseteq K(S)$ and $\sigma \in H$. Write $t = \sigma \in \{0, \dots, 7\}$, $B = S \cap H$, and $m = |B|$. For $u \in \{0, \dots, 7\}$, define

$$C_u = \{a \in V : \text{the first three coordinates of } a \text{ equal } u\}, \qquad \mu_u = \sum_{h \in B} (-1)^{u \cdot h}.$$

The sets C_u are the cosets of $H^\perp$; each has 16 elements. In this formula $u \cdot h$ is the three-coordinate binary dot product.

Lemma 4.1. *The following hold:*

1. *If $u \neq w$, no eigenvalue value occurs both on C_u and on C_w.*
2. *For $a \in C_u$, $-n \leq \lambda_a \leq n$ and $\lambda_a \equiv n - 2(u \cdot t) \pmod 4$. Also $\lambda_0 = n$.*
3. *The coset sums and total second moment are*

$$\sum_{a \in C_u} \lambda_a = 16\mu_u, \qquad \sum_{a \in V} \lambda_a^2 = 128n. \tag{4}$$

In particular, $\mu_0 = m$.

Proof. If $u \neq w$, some $e_i \in H$, $1 \leq i \leq 3$, has different dot products with $a \in C_u$ and $b \in C_w$. Since $e_i \in K(S)$, their eigenvalues cannot be equal. The range and trivial eigenvalue follow from (1); the congruence is (3), because $\sigma = t \in H$.

For the coset sum, write $a = (u, v)$ and $s = (h, y)$ according to $V = \mathbb{F}_2^3 \oplus \mathbb{F}_2^4$. Character orthogonality gives

$$\sum_{a \in C_u} \lambda_a = \sum_{(h,y) \in S} (-1)^{u \cdot h} \sum_{v \in \mathbb{F}_2^4} (-1)^{v \cdot y} = 16 \sum_{h \in B} (-1)^{u \cdot h}.$$

Similarly,

$$\sum_{a \in V} \lambda_a^2 = \sum_{s, s' \in S} \sum_{a \in V} (-1)^{a \cdot (s + s')} = 128|S|,$$

since the inner sum is nonzero precisely when $s = s'$. □

For a coset other than C_0, the mean of its 16 eigenvalues is μ_u. On C_0 we separate out $\lambda_0 = n$; the other 15 values have mean $(16m - n)/15$. This distinction retains the contribution of the trivial character exactly.

Lemma 4.2 (Two-point second-moment bound)**.** *Let $x_1, \dots, x_N$ be real numbers with $N \geq 1$ and mean M. Let α be the greatest of these numbers at most M, and let β be the least of them at least M. Then*

$$\sum_{i=1}^{N} x_i^2 \geq N\big(M^{\,2} + (M - \alpha)(\beta - M)\big). \tag{5}$$

Proof. Both endpoints exist by the definition of a mean. No x_i lies strictly between them, so $(x_i - \alpha)(x_i - \beta) \geq 0$. Summing and using $\sum_i x_i = NM$ gives

$$\sum_i x_i^2 \geq N\big((\alpha + \beta)M - \alpha\beta\big) = N\big(M^{\,2} + (M - \alpha)(\beta - M)\big).$$

This includes $\alpha = \beta = M$. □

5 The finite inequality

We state a finite relaxation of the spectral restrictions and show that every hypothetical large class supplies a feasible instance. A *case* is a triple (n, B, t) with

$$0 \le n \le 127, \quad B \subseteq \{1, \dots, 7\}, \quad |B| \le n, \quad t \in \{0, \dots, 7\}. \tag{6}$$

Set $m = |B|$, and, for $u = 0, \dots, 7$, set

$$M_0 = \frac{16m - n}{15}, \qquad M_u = \mu_u = \sum_{h \in B} (-1)^{u \cdot h} \ (u \neq 0), \qquad r_u \equiv n - 2(u \cdot t) \pmod 4. \tag{7}$$

The residue r_u is represented by one of $0, 1, 2, 3$.

Definition 5.1. An *admissible bracket assignment* for a case (n, B, t) consists of eight pairs of integers (α_u, β_u), $0 \le u \le 7$, satisfying:

1. $-n \le \alpha_u \le M_u \le \beta_u \le n$;
2. if $\alpha_u = \beta_u$, then $\alpha_u = M_u$;
3. $\alpha_u \equiv \beta_u \equiv r_u \pmod 4$;
4. if $u \neq 0$, neither endpoint is n;
5. the sets $\{\alpha_u, \beta_u\}$ are pairwise disjoint as u varies.

Its *cost* is $\sum_{u=0}^{7} c_u$, where

$$c_0 = n^2 + (\alpha_0 + \beta_0)(16m - n) - 15\alpha_0\beta_0, \tag{8}$$

$$c_u = 16\big((\alpha_u + \beta_u)\mu_u - \alpha_u\beta_u\big) \quad (u \neq 0). \tag{9}$$

The cost for $u \neq 0$ is the right side of (5) with $N = 16$ and $M = \mu_u$. For $u = 0$ it is n^2 plus that bound with $N = 15$ and $M = M_0$. Although M_0 can be rational, the expanded costs (8)–(9) are integers. A *bracket* means one of these integer pairs. An endpoint equal to the mean is allowed even when the other endpoint is different.

Lemma 5.2 (Extraction)**.** *If $H \subseteq K(S)$ and $\sigma \in H$, the case $(|S|, S \cap H, \sigma)$ has an admissible bracket assignment of cost at most $128|S|$.*

Proof. For $u \neq 0$, apply Lemma 4.2 to the 16 values on C_u; for $u = 0$, apply it to the 15 values on $C_0 \setminus \{0\}$. Their means are exactly (7) by Lemma 4.1. Take the resulting observed endpoints α_u, β_u. The range, the equality case, and the residues give conditions 1–3. Condition 4 holds because $n = \lambda_0$ occurs on C_0 and cannot occur on another coset. The same separation gives condition 5. Adding the eight bounds, with the separate term n^2 on C_0, and applying (4) gives the claimed cost bound. □

Lemma 5.3 (Finite inequality)**.** *For every case (6), every admissible bracket assignment has cost strictly greater than $128n$.*

Section 6 proves this finite assertion using exact certificates.

6 Weak duality and the finite computation

We prove a lower bound that reduces Lemma 5.3 to small, independent integer calculations. For a fixed case, let J_u be the set of all brackets for coset u satisfying conditions 1–4 of Definition 5.1. At this stage disjointness between cosets is not imposed. For $j = (\alpha, \beta) \in J_u$, let $c_u(j)$ be its cost from (8) or (9).

Lemma 6.1 (Penalty bound)**.** *Assign nonnegative rational weights w_v to integers $v \in [-n, n]$. For a bracket j, let $w(j)$ be the sum of the weights on its distinct endpoints, counting a repeated endpoint once. If every J_u is nonempty, every admissible assignment has cost at least*

$$L(w) = \sum_{u=0}^{7} \min_{j \in J_u} \bigl(c_u(j) + w(j)\bigr) - \sum_{v=-n}^{n} w_v. \tag{10}$$

Proof. For the chosen bracket j_u of each coset,

$$c_u(j_u) \geq \min_{j \in J_u} (c_u(j) + w(j)) - w(j_u).$$

By disjointness, any integer is an endpoint in at most one chosen bracket. Nonnegativity therefore gives $\sum_u w(j_u) \leq \sum_v w_v$. Summing the preceding inequalities proves (10). □

This is weak duality for the linear programming relaxation in which one bracket is chosen per coset and each endpoint value has capacity one. No linear programming solver is part of the verification: nonnegative weights with $L(w) > 128n$ suffice. If some J_u is empty, there can be no admissible assignment and no weights are needed.

6.1 The finite computation

We now specify the finite arithmetic assertion used to finish the proof. A case can share its certificate with any case having the same *case key*

$$(n, m, \text{the sorted list of } (\mu_u, r_u) \text{ for } 1 \leq u \leq 7)\,. \tag{11}$$

These data determine J_0 and its cost, and determine the other seven bracket lists and costs up to permutation. Such a permutation preserves disjointness and total cost.

There are

$$8 \sum_{B \subseteq \{1,\ldots,7\}} (128 - |B|) = 8(128 \cdot 128 - 7 \cdot 64) = 127{,}488$$

cases. Forming (11) for every case (6) gives 3,984 distinct keys.

A *certificate entry* consists of a case key together with either an assertion that a bracket list is empty or nonnegative integer weights W_v representing $w_v = W_v/15000$. Unlisted weights are zero. For a weight entry, the required inequality is

$$\sum_{u=0}^{7} \min_{(\alpha,\beta) \in J_u} \bigl(15000 c_u(\alpha, \beta) + W_\alpha + \mathbf{1}_{\beta \neq \alpha} W_\beta\bigr) - \sum_v W_v > 1{,}920{,}000n. \tag{12}$$

Here $\mathbf{1}_{\beta \neq \alpha}$ is one when the endpoints differ and zero when they agree. All operations are integer operations. The sets J_u are formed by ranging α, β over the integers in $[-n, n]$ and applying conditions 1–4; comparisons with M_0 are made after multiplication by 15.

The finite assertion established by the supplied data is that every entry has either a genuinely empty J_u or nonnegative weights satisfying (12), and that the entries cover all cases. To *cover* means that every case's key occurs among the entries with the correct coset data. Both parts of this assertion are checked exactly, in Lean and by the separate integer checker described in Section 8. The certificate weights were obtained from a floating-point linear programming dual and converted to rational numbers. The complete weight tables accompany the paper; Appendix A gives a worked instance and the routine certificate counts.

Proof of Lemma 5.3. Each case is covered by a checked entry. An empty bracket list excludes an admissible assignment. Otherwise, the checked nonnegative weights satisfy (12), which is $15000L(w) > 15000 \cdot 128n$. Lemma 6.1 therefore puts every admissible assignment's cost strictly above $128n$. □

Proof of the upper bound in Theorem 1.1. If a pairwise strongly cospectral set has more than four vertices, Lemmas 3.1 and 3.3 put its connection set in the position required by Lemma 5.2. That lemma produces an admissible assignment of cost at most $128n$, contradicting Lemma 5.3. □

7 The connected example attaining four

We use the instance of Árnadóttir–Godsil's odd-dimensional construction [1, Theorem 10.2] that proves attainment. With the standard basis $e_1, \dots, e_7$, let

$$\begin{aligned} S = {} & \{e_i : 1 \le i \le 7\} \cup \{e_1 + e_2, e_1 + e_3, e_2 + e_3\} \\ & \cup \{e_i + e_j : 4 \le i < j \le 7\} \cup \{e_1 + e_2 + e_3 + e_i : 4 \le i \le 7\}. \end{aligned}$$

These four disjoint parts have sizes $7, 3, 6, 4$, so the graph is 20-regular. The connection set contains a basis, so the graph is connected. In the integer encoding above it is

$$S = \{1, 2, 3, 4, 5, 6, 8, 15, 16, 23, 24, 32, 39, 40, 48, 64, 71, 72, 80, 96\}.$$

For this set, the character criterion gives

$$K(S) = \{0, 7, 120, 127\} = \langle e_1 + e_2 + e_3,\ e_4 + e_5 + e_6 + e_7 \rangle. \tag{13}$$

Figure 1 shows this connected example and highlights the four vertices in (13). Its character eigenvalue fibers provide a short explicit check of the four claimed vertices: Table 1 records the sign of $\chi_a(g)$ throughout each fiber, for the three nonzero g in (13). The sign at $g = 0$ is always positive.

For each displayed g, the value $\chi_a(g)$ is therefore constant as a ranges over an eigenvalue fiber, as required by Lemma 3.1. The upper bound already proved excludes any further vertices from this class. This establishes attainment and completes the proof of Theorem 1.1.

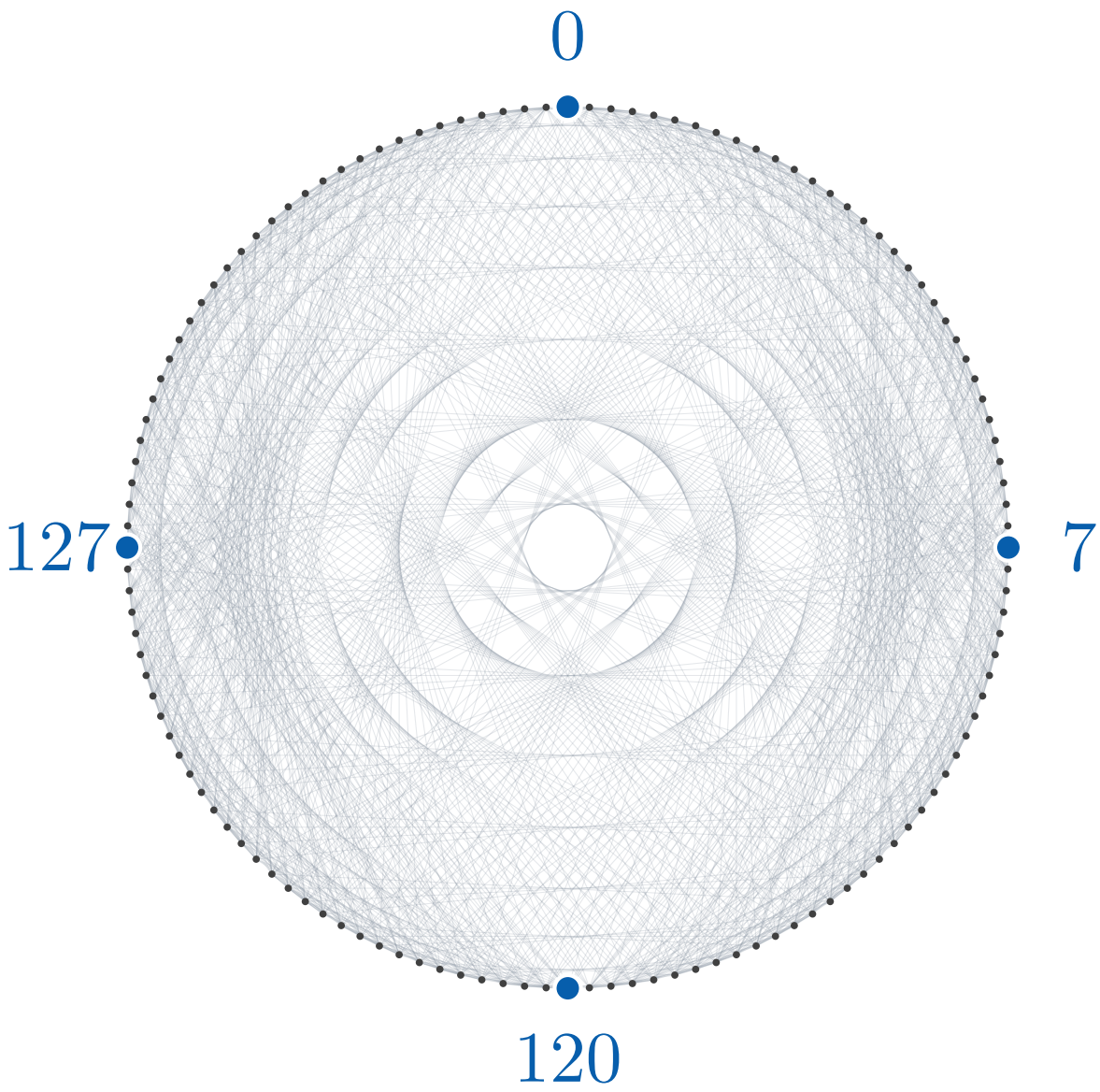


Figure 1: The connected 20-regular cubelike graph of Árnadóttir and Godsil on 128 vertices. All edges are shown in gray; the four vertices $0, 7, 120, 127$ of one strongly cospectral class are highlighted. The circular layout places these four vertices at the cardinal points to show the class within the full graph. The drawing is generated from the displayed connection set.

Eigenvalue θ	Multiplicity	$g = 7$	$g = 120$	$g = 127$
20	1	+	+	+
12	3	+	+	+
10	4	+	−	−
6	8	−	+	−
4	7	+	+	+
2	16	+	−	−
0	32	−	−	+
−2	24	−	+	−
−4	21	+	+	+
−6	12	+	−	−

Table 1: Character signs on the eigenvalue fibers of the connected example. Each sign is constant over the entire indicated fiber.

8 Formal verification

The universal upper bound is formalized in Lean and mathlib. The formal theorem is named `CubelikeSC.endpoint_sc`:

```
theorem endpoint_sc : EndpointSC
```

The source states its expanded quantifiers; the displayed form uses the definition `EndpointSC`, the universal statement

$$\forall S \subseteq V, \quad 0 \notin S \implies \forall P \subseteq V, \quad (\forall u, v \in P,\ u, v \text{ are strongly cospectral}) \implies |P| \leq 4.$$

Here the formal vertex type is $(\mathbb{Z}/2\mathbb{Z})^7$; S and P are finite sets, as are all subsets of this vertex set. The adjacency convention $x - y \in S$ agrees with $x + y \in S$ over $\mathbb{F}_2$. Strong cospectrality is defined using orthogonal projections onto the real adjacency eigenspaces: for each real eigenvalue θ, the projections of the two coordinate vectors must agree or be negatives of each other. The sign can depend on θ. Thus the formal assertion is precisely the upper bound of Theorem 1.1, with no connectivity hypothesis. The formal proof includes the spectral reduction, the exact inequalities on the supplied weights and coverage of every case. Independent integer calculations check the same inequalities directly for every original case.

The converse character criterion and the attaining construction remain outside Lean. The written argument and the published construction give the lower bound, with independent exact computations of the character sums and adjacency spectral projections. The accompanying checking guide gives reproduction instructions, full formal definitions and logical dependencies.

Appendix A Certificate details

A.1 A worked certificate

The case $n = 12$, $B = \{1, 2\}$, $t = 4$ illustrates how the weights exclude an assignment. In unscaled cost units, its certificate has the nonzero weights

v	-8	-6	-4	-2	0	2	4	6	8
w_v	256	384	896	1024	1984	1024	1216	384	704

.

The integer weights in the data file are 15000 times these values. In coset order $u = 0, \dots, 7$, the means and residues are

$$(M_u) = (4/3, 0, 0, -2, 2, 0, 0, -2), \qquad (r_u) = (0, 0, 0, 0, 2, 2, 2, 2).$$

Enumerating the permitted pairs gives

$$\left(\min_{j \in J_u}(c_u(j) + w(j))\right)_{u=0}^{7} = (1920, 1984, 1984, 1984, 1088, 1344, 1344, 1088),$$
$$\sum_v w_v = 7872.$$

For example, a minimizing pair on the zero coset is $(-12, 4)$. Its cost is

$$12^2 + (-12 + 4)(32 - 12) - 15(-12)(4) = 704,$$

so its adjusted cost is $704 + w_{-12} + w_4 = 704 + 0 + 1216 = 1920$. Minimizing pairs on the other cosets, in order, are $(-12, 0), (-12, 0), (-8, 8), (-10, 2), (-10, 2), (-10, 2), (-10, -2)$. Thus (10) gives

$$L(w) = 12736 - 7872 = 4864 > 1536 = 128 \cdot 12.$$

Without endpoint penalties, the sum of the eight individual minimum costs is only 608. The disjointness requirement, expressed through the penalties, is essential to this certificate.

The resulting certificate counts are summarized in Table 2. The 58 empty-list entries have

Certificate disposition	Number of case keys
A coset has no allowed bracket	58
Nonnegative weights satisfy (12)	3,926
Total	3,984

Table 2: Exact certificate outcomes. Each key can represent several cases; these are counts of keys, not counts of connection sets or cases.

$n = 0, 1, 2, 3, 4, 5, 6$, with respectively $2, 5, 9, 15, 16, 9, 2$ keys. Together they represent 1,628 original cases. For each entry, forming the relevant J_u verifies that no integer pair satisfies conditions 1–4. For example, at $n = 0$ the only possible endpoint is zero, which condition 4 forbids on a nonzero coset. A separate exact Python checker forms each case's own eight bracket lists and checks the matching certificate on those lists, without assuming the key-sharing argument.

Acknowledgments

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