The strong Lefschetz property for complex reflection coinvariants

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Abstract

The ordinary coinvariant algebra over $\mathbb{C}$ of every finite complex reflection group has the strong Lefschetz property: one linear element has maximal-rank multiplication by every power in every degree. We give a computer-assisted proof for G_{34}, the exceptional type excluded by Wachi's earlier results; the other types follow from the work of Harima–Watanabe, Harima–Migliore–Nagel–Watanabe, McDaniel, and Wachi. The question is recorded in Wachi's 2013 conjecture on complex reflection coinvariants. The proof uses the Conway–Sloane orbit invariants in the coordinates developed by Oura–Sekiguchi. Exact finite-field polynomial certificates, specialization, and a Hilbert-function comparison establish the strong Lefschetz property for a relative coinvariant algebra. The free-extension theorem then transfers it to the ordinary coinvariant algebra, using Wachi's result for G_{33}.

This work has not yet been reviewed by experts. The proofs, the certificate checkers and the text were produced by AI models (from OpenAI and Anthropic), working under the author's direction.

1 Introduction

Let V be a nonzero finite-dimensional complex vector space, and let $W \subseteq \mathrm{GL}(V)$ be a finite group generated by complex reflections. A complex reflection is a nonidentity linear transformation fixing a hyperplane pointwise. Give $R = \mathrm{Sym}(V^*)$ its ordinary polynomial grading, and let R_+^W denote the positive-degree invariants. The *coinvariant algebra* is

$$E_W = R/(R_+^W R).$$

For example, the cyclic reflection group of order m in one variable has coinvariant algebra $\mathbb{C}[x]/(x^m)$. Multiplication by each power of x has the largest possible rank in every degree.

Definition 1.1. A finite-dimensional graded algebra $D = \bigoplus_{i\geq 0} D_i$ has the *strong Lefschetz property* if there is an element $\ell \in D_1$ such that

$$\mathrm{rank}(\ell^k : D_i \longrightarrow D_{i+k}) = \min\{\dim D_i, \dim D_{i+k}\} \qquad (i, k \geq 0).$$

Such an element is a strong Lefschetz element. Graded algebras below need not be generated by their degree-one pieces.

Theorem 1.2. *For every irreducible finite complex reflection group W on a nonzero complex vector space, the ordinary coinvariant algebra E_W has the strong Lefschetz property.*

The product reduction in Corollary 5.3 extends this statement to reducible groups and representations with fixed directions.

The contribution here is the G_{34} case. Wachi proved the other exceptional nonreal cases of rank at least three, including G_{33} [21, 22]; Wachi's author statement explicitly excludes G_{34}. We use that G_{33} result as a literature premise. The precise source qualification is stated in Section 6. McDaniel and Smith jointly announced the Bott–Samelson embedding work in 2016 [1, pp. 5, 10–11]. Smith's talk asserted an SLP deduction for groups generated by order-two reflections, a class that includes G_{34} [1, p. 6]. The same report records McDaniel's warning about inheritance [1, McDaniel's talk, p. 5]. McDaniel–Smith [12, Proposition 5.9] exhibit a type-A_3 Bott–Samelson map without the inheritance property, although both rings have SLP. Thus the embedding alone does not supply the required multiplication ranks. This work supplies those ranks for G_{34} by relative-coinvariant certificates and free extension, rather than by inheritance along a Bott–Samelson embedding. The cited noninheritance example does not disprove the announced SLP conclusion for G_{34}. The report also records a McDaniel–Smith–Wachi breakout lemma giving sufficient conditions for SLP of the base in a Leray–Hirsch decomposition, and Smith's use of it for G_{25} and G_{26} [1, §4.4, p. 10]. This is an earlier base-SLP strategy close to the free-extension step here; Wachi's 2013 Lemma 36 already treats G_{24} through G_{27} [18]. The difficulty is to obtain all multiplication ranks, rather than only a nonzero highest power of a linear form. Instead of treating the full G_{34} coinvariant algebra directly, we prove the required ranks on a smaller relative algebra through an auxiliary invariant quotient. The free-extension method is due to Harima–Watanabe and McDaniel–Chen–Iarrobino–Macias Marques [8, 11]. The orbit invariants are due to Conway–Sloane [5], and their useful coordinate descriptions and subgroup model are supplied by Oura–Sekiguchi [13].

Earlier cases and the question

Wachi states Conjecture 37 in the proceedings of Wachi's November 29, 2013 lecture [18, Conjecture 37, proceedings p. 155, PDF p. 16], [19]. The original Japanese sentence is:

> 複素鏡映群の余不変式環は、強レフシェッツ性を持つ。

Translation: "The coinvariant ring of a complex reflection group has the strong Lefschetz property." The sentence does not restrict the group to be irreducible. Theorem 1.2 treats the irreducible factors; Corollary 5.3 supplies the full assertion. This is the earliest dated public presentation identified here; the original online upload date of the notes is not established. Those notes already list the five remaining cases $G_{29}, G_{31}, G_{32}, G_{33}, G_{34}$. Wachi's March 2014 abstract [20, Functional Analysis, talk 12, p. 78], in the March 3 collection, also records the question with five exceptional types remaining. The question is thus dated to its public 2013 presentation, rather than to the 2025 article. The Shephard–Todd classification [14] reduces Theorem 1.2 to the cases in Table 1. Wachi's lecture, Theorem 4 and Tables 1–3, is also a convenient source for this classification and the degrees used below.

For clarity, the monomial-family citation concerns the ordinary coinvariant algebra in the standard grading. Put $s = r/p$ and let e_i be the ith elementary symmetric polynomial. Its presentation is

$$\frac{\mathbb{C}[x_1, \dots, x_n]}{(e_1(x_1^r, \dots, x_n^r), \dots, e_{n-1}(x_1^r, \dots, x_n^r), (x_1 \cdots x_n)^s)}. \tag{1}$$

To see this, the diagonal ratios in $G(r, p, n)$ force invariant monomial exponents to coincide modulo r, and the allowed single-coordinate diagonal elements force that common residue to be a multiple of s. After extracting a power of $t = (x_1 \cdots x_n)^s$, permutation invariance gives a symmetric polynomial in x_i^r. Its last elementary symmetric generator is t^p. Thus the invariant ring is generated by

Table 1: Earlier results used for types other than G_{34}.

Types	Result and exact location
Cyclic groups	The one-variable example above, including the trivial rank-one quotient.
Type A and real reflection groups	McDaniel [10, Theorem 1]; the proof in §5 includes H_3 and H_4. Extension of scalars from $\mathbb{R}$ to $\mathbb{C}$ preserves multiplication ranks.
$G(r,p,n)$, $r \geq 2$, $p \mid r$	Harima–Watanabe [7, Notation and Remark 6.3, Examples 6.4 and 6.8], with $s = r/p$; the invariant presentation is explained below.
$G_4, \ldots, G_{22}$	These have rank two. Harima–Migliore–Nagel–Watanabe [6, Proposition 4.4] prove SLP for homogeneous Artinian quotients in two variables in characteristic zero.
G_{23}, G_{28}, G_{30}, G_{35}, G_{36}, G_{37}	These are $H_3, F_4, H_4, E_6, E_7, E_8$, respectively, and are included in McDaniel's real-group theorem.
$G_{24}, \ldots, G_{27}$, $G_{29}, G_{31}, G_{32}, G_{33}$	Wachi [21, 22]. Lecture Lemma 36 already treats the first four; the author repository lists all eight proved types and leaves G_{34}.

$e_1, \ldots, e_{n-1}, t$; these are algebraically independent since the polynomial ring is integral over its invariant subring and has dimension n. This proves (1). For $p = 1$, Example 6.4 of [7] applies with $s = r$; for $p > 1$, Example 6.8 applies with $0 < s < r$. Rank one is immediate. The case $r = 1$ is type A, with its fixed summand killed by its linear invariant.

For Weyl groups, McDaniel's introduction uses Borel's identification with flag-variety cohomology and the hard Lefschetz theorem [10, Introduction]. Stanley's earlier work applies hard Lefschetz to Weyl-group order structures [15]. McDaniel also credits Maeno–Numata–Wachi and Numata–Wachi for the noncrystallographic cases; see [10, Introduction and §5]. Yoshida characterizes Lefschetz elements for complex reflection groups containing a binary polyhedral subgroup of index two [24]; this related rank-two work is not needed for the existence theorem supplied by [6]. Related work of McDaniel–Smith [12] studies embeddings into Bott–Samelson algebras. Such an embedding alone does not transfer SLP: their Proposition 5.9 exhibits the missing inheritance condition even in type A_3. The argument here instead supplies SLP of both the base and fiber of a free extension. It asserts neither an inheritance theorem for arbitrary subalgebras nor an implication from maximal height to SLP.

Proof overview

Use the subgroup chain

$$H = G(3,3,4) \ \subset \ K = G_{33} \ \subset \ W = G_{34}.$$

The subgroup K fixes the linear form $z = (x_5 - x_6)/2$. Let A be the relative coinvariant algebra for $K \subset W$. The ordinary W coinvariant algebra is a free extension of A with the ordinary K coinvariant algebra as fiber. The latter has SLP by Wachi. It remains to prove that z is a strong Lefschetz element of A.

The invariant ring of H is an explicit polynomial ring. Its quotient by the positive W-invariants, denoted C, is a graded free A-module with a known fiber B. For every k, this makes the Hilbert function of $C/(z^k)$ at least the function predicted by maximal-rank multiplication on A, multiplied by the Hilbert series of B. Exact polynomial identities over $\mathbb{F}_{1009}$ give a monomial-ideal upper bound. A finite monomial enumeration shows that the upper and lower bounds agree for every required k. Specialization goes in the needed direction because nonzero minors modulo 1009 remain nonzero in characteristic zero.

Section 2 defines the model and its integral coordinates. Section 3 states the finite facts certified by the programs. Sections 4 and 5 prove the bridges from those facts to SLP over $\mathbb{C}$. The final section describes the certificate format, reproduction, and the trusted boundary.

2 The group model and integral invariants

This section fixes the characteristic-zero objects to which the modular inputs must belong. Let $\omega = e^{2\pi i/3}$ and $\theta = 1 + 2\omega = \omega - \omega^2$. The following vectors in $\mathbb{Z}[\omega]^6$ form a set $\mathcal{V}$:

$$\begin{aligned} &\pm\,\theta(\omega^a e_i - \omega^b e_j), && 1 \le i < j \le 6, \quad a, b \in \{0,1,2\}, \\ &\pm\,(\omega^{a_1}, \dots, \omega^{a_6}), && a_i \in \{0,1,2\}, \quad \sum_i a_i \equiv 0 \pmod 3. \end{aligned}$$

Here e_i is the ith coordinate vector. The two sets have 270 and 486 elements, respectively. This is the Conway–Sloane construction [5, Theorem 10] in the form described in [13, §2]. For a vector $n \in \mathcal{V}$, its Hermitian squared norm is 6, and the corresponding reflection is

$$s_n(u) = u - \frac{\overline{n}^T u}{3}\, n.$$

In the notation of Oura–Sekiguchi, use the normals

	P_1	P_2	P_3	Q_1	R_1	R_2
n	$\theta(e_2 - e_3)$	$\theta(e_3 - e_4)$	$\theta(e_4 - e_5)$	$\theta(e_1 - e_2)$	$\theta(e_1 - \omega^2 e_2)$	$(1,1,1,1,1,1)$.

Let W be generated by all six reflections, let $K = \langle P_1, P_2, Q_1, R_1, R_2\rangle$, and let $H = \langle P_1, P_2, Q_1, R_1\rangle$. The identification of W with G_{34} and K with G_{33} is [13, §§2,6.1]; H acts as $G(3,3,4)$ on the first four coordinates and fixes the last two. The invariant degrees are

$$\begin{array}{c|l} W & (6,12,18,24,30,42) \\ K & (1,4,6,10,12,18) \\ H & (3,4,6,9,1,1). \end{array} \tag{2}$$

These are literature premises, also listed in [18, Tables 2–3]. The degree-one invariant of K accounts for its fixed line.

The exact group check verifies preservation of $\mathcal{V}$, the reflection matrices, inclusions, and orders

$$|W| = 39\,191\,040, \qquad |K| = 51\,840, \qquad |H| = 648. \tag{3}$$

Its permutation action on $\mathcal{V}$ is faithful: six vectors in that set have a determinant with nonzero norm 8748 and hence span. The subgroup K fixes $e_5 - e_6$; unitarity implies that it fixes its dual coordinate z. The checker verifies these assertions by exact Eisenstein-integer arithmetic and a faithful finite permutation representation. Agreement of orders by itself is not used to identify the named groups or their invariant degrees.

Put $y = (x_5 + x_6)/2$, $z = (x_5 - x_6)/2$, and define

$$a = \sum_{i=1}^{4} x_i^3, \qquad b = x_1x_2x_3x_4, \qquad c = \sum_{i=1}^{4} x_i^6, \qquad d = \sum_{i=1}^{4} x_i^9.$$

Their weights, in the order a, b, c, d, y, z, are $(3,4,6,9,1,1)$. Let

$$D = \left(\mathbb{Z}[\omega, 1/42!]\right)_{(1009,\,\omega-374)}, \qquad L = \mathrm{Frac}(D) = \mathbb{Q}(\omega).$$

The integer 1009 is prime, and $374^2 + 374 + 1 = 139 \cdot 1009$. Thus D has the residue map $\omega \mapsto 374$ to $\mathbb{F}_{1009}$. All denominators through 42! are units.

Lemma 2.1. *The invariant ring of H over D is $D[x_1,\dots,x_6]^H = D[a,b,c,d,y,z]$, a polynomial ring with the weights above. The same assertion holds over $\mathbb{F}_{1009}$ and over L.*

Proof. The diagonal cube-root ratios force the first four exponents of an invariant monomial to coincide modulo three. The differences of distinct cube roots are units in D, so this follows coefficient by coefficient. Every such monomial has a unique expression as b^r times a monomial in $x_1^3,\dots,x_4^3$, with $0 \le r < 3$. The permutation subgroup fixes b and permutes these four cubes. The fundamental theorem of symmetric polynomials over $\mathbb{Z}$ therefore gives the elementary generators e_1,e_2,e_3,e_4 in the cubes, with $e_4 = b^3$. Their first three values are

$$e_1 = a, \qquad e_2 = (a^2 - c)/2, \qquad e_3 = (a^3 - 3ac + 2d)/6.$$

This proves generation by a,b,c,d. For independence, split a purported polynomial relation by the exponent of b modulo three. The three summands have disjoint exponent residues in the original variables. Each would give a relation in the independent elementary symmetric functions e_1,e_2,e_3,e_4, so all vanish. Finally, the change $x_5 = y+z$, $x_6 = y-z$ is invertible since 2 is a unit. The same argument works in the residue field and in L. □

For $q \in \{6,12,18,24,30,42\}$, define the unscaled orbit sum

$$f_q(x) = \sum_{v \in \mathcal{V}} (v^T x)^q. \tag{4}$$

The group check proves invariance: $\mathcal{V}$ is stable under the generators and under conjugation, hence also under their transposes and inverses. Thus either equivalent dual-action convention preserves the sums. By Lemma 2.1, each f_q has a unique expression $F_q \in D[a,b,c,d,y,z]$. Denote its reduction modulo the maximal ideal by I_q, and put

$$S = \mathbb{F}_{1009}[a,b,c,d,y,z], \qquad I = (I_6, I_{12}, I_{18}, I_{24}, I_{30}, I_{42}).$$

The committed modular inputs are checked against (4) by an exhaustive coefficient calculation. For the second vector family, its character sum is 486 when all six exponent residues modulo three agree and zero otherwise. The checker tests all 3^6 residue tuples and expands every possible surviving exponent tuple; it sums the first family on each two-coordinate support. It then substitutes $x_5 = y+z$, $x_6 = y-z$ and compares all coefficients with the substituted I_q. Multinomial coefficients use factorials at most 42. Lemma 2.1 makes the modular substitution injective, so this proves that the six inputs are the stated reductions, rather than a presentation fitted by point evaluations.

3 The finite certificate facts

This section gives the exact finite assertions used by the proof. Write $H_D(t) = \sum_j \dim(D_j) t^j$ for the Hilbert series of a finite graded algebra over a field. Introduce the following explicit integer polynomials:

$$h(t) = \frac{(1-t^{24})(1-t^{30})(1-t^{42})}{(1-t)(1-t^4)(1-t^{10})} = \sum_{j=0}^{81} h_j t^j, \tag{5}$$

$$b(t) = \frac{(1-t^{10})(1-t^{12})(1-t^{18})}{(1-t)(1-t^3)(1-t^9)}, \tag{6}$$

$$Q_k(t) = \sum_{j \ge 0} \max(0, h_j - h_{j-k}) t^j, \qquad h_j = 0 \text{ for } j \notin [0,81]. \tag{7}$$

Their meaning as characteristic-zero Hilbert series will be established in Section 5.

Use weighted degree reverse lexicographic order on S, with variables a, b, c, d, y, z in that order and z last. The certificate gives homogeneous polynomials $G_1, \dots, G_{534}$ and polynomials T_{qj}. Write $\mathrm{LT}(g)$ for the leading monomial of a nonzero polynomial g, and $\mathrm{in}(U)$ for the monomial ideal generated by the leading monomials of all nonzero elements of an ideal U. Let $J = (\mathrm{LT}(G_1), \dots, \mathrm{LT}(G_{534}))$.

Proposition 3.1 (Certified finite facts)**.** *The six modular inputs defined above and the certificate satisfy:*

1. *Every G_j is nonzero and $G_j = \sum_{q\in\{6,12,18,24,30,42\}} I_q T_{qj}$ coefficient by coefficient. The declared leading exponents agree with those reconstructed from the polynomials and the specified order.*
2. *The monomial ideal J contains the pure powers $a^2, b^3, c^3, d^4, y^{68}, z^{82}$. Its complement consists of exactly 60 480 monomials, all of weighted degree at most 108.*
3. *For every $1 \le k \le 81$, the complete Hilbert series satisfies*

$$H_{S/(J,z^k)}(t) = Q_k(t)b(t).$$

These are finite exact assertions, verified by the checkers distributed with this paper. Two arithmetic implementations verify part (1). Two separate monomial enumerations verify parts (2)–(3), using the same checked leading exponents but separate traversal methods. There is a finite bound before any Hilbert-series comparison: each standard exponent is below the corresponding pure-power exponent. Every monomial outside that box lies in J. Thus enumeration proves the absence of a tail as well as agreement of the displayed coefficients; the degree 108 bound is an output of the complete cover, not an assumed stopping point.

For one enumeration, every standard monomial is uniquely uz^i, where z does not divide u and $0 \le i < L_u$. The quotient by z^k has the basis with $0 \le i < \min(k, L_u)$. A separate enumeration traverses the entire bounded standard set directly. Both calculate (7) by integer coefficient arithmetic. Further checks certify a Gröbner basis and the converse inclusion of ideals, but only the identities and the monomial upper bound in Proposition 3.1 are needed below.

4 From the residue field to the invariant ideal

We now justify the characteristic-zero upper bounds and prove that the chosen orbit sums generate the whole W-invariant ideal. These steps do not assume flatness or an integral presentation for R^K.

Lemma 4.1 (Specialization)**.** *Put $\widetilde{C} = L[a, b, c, d, y, z]/(F_q : q \in \{6, 12, 18, 24, 30, 42\})$. For every $j \ge 0$ and $k \ge 1$,*

$$\dim_L(\widetilde{C}/(z^k))_j \le \dim_{\mathbb{F}_{1009}}(S/(I, z^k))_j \le \dim_{\mathbb{F}_{1009}}(S/(J, z^k))_j.$$

The same inequalities hold without adjoining z^k.

Proof. In weighted degree j, list the finitely many monomials. Make a matrix over D whose columns are all products of the F_q and of z^k with monomials of the complementary degree. Its cokernel is the degree-j quotient. Every minor nonzero in the residue field is already nonzero over L. Hence the characteristic-zero rank is at least the residue rank, which proves the first inequality. This does not lift a modular zero relation.

By Proposition 3.1(1), $G_j \in I$, so $J \subseteq \mathrm{in}(I)$. Also z^k is itself a leading monomial in $I + (z^k)$. Therefore $J + (z^k) \subseteq \mathrm{in}(I + (z^k))$. The standard monomials of the initial ideal are a vector-space basis of the corresponding quotient, which gives the second inequality. This basis fact follows directly by leading-term reduction: nonstandard monomials reduce to smaller ones, and a nontrivial relation among standard monomials would have a leading term in the initial ideal. Positivity of the weights makes the order well-founded. No equality of the initial ideals is needed. □

Proposition 4.2. *Over $\mathbb{C}$, the orbit sums generate the invariant ring:*

$$R^W = \mathbb{C}[f_6, f_{12}, f_{18}, f_{24}, f_{30}, f_{42}], \qquad R = \mathbb{C}[x_1, \dots, x_6].$$

Proof. Proposition 3.1(2) makes S/J finite-dimensional. Lemma 4.1 without z^k makes $\widetilde{C}$ finite-dimensional. After extending scalars to $\mathbb{C}$ and using Lemma 2.1, this says that $R^H/(f_q)$ is finite-dimensional. The ring R is finite over R^H: each element is integral by its finite orbit polynomial, and R is generated by finitely many integral elements. Consequently $R/(f_q)$ is finite as well.

Thus the six f_q form a homogeneous system of parameters of the six-variable polynomial ring. A polynomial ring is Cohen–Macaulay, and such a parameter system is a regular sequence; see [17, Tags 00ND and 02JN]. Apply the local statement at the homogeneous maximal ideal; a nonzero graded kernel remains nonzero there. Regularity gives

$$H_{R/(f_q)}(t) = \frac{\prod_{q\in\{6,12,18,24,30,42\}}(1-t^q)}{(1-t)^6}, \qquad \dim R/(f_q) = \prod_{q\in\{6,12,18,24,30,42\}} q = 39\,191\,040.$$

Let $P = \mathbb{C}[f_q : q \in \{6, 12, 18, 24, 30, 42\}]$. Homogeneous lifts of a vector-space basis of $R/(f_q)$ generate R over P, by induction on degree: subtract the corresponding lift, express the remainder using the positive-degree f_q, and apply the induction to its coefficients. Thus R is finite over P. Equal dimensions show that the six generators of P are algebraically independent. Indeed, the kernel of the map from a polynomial ring in six variables would otherwise be a nonzero prime and lower the dimension.

The same lifts give a surjection from a graded free P-module onto R. The displayed regular-sequence Hilbert series shows that the source and target have equal Hilbert series, so its kernel is zero. Its rank is $\prod q = |W|$, by (3), and therefore $[\mathrm{Frac}\, R : \mathrm{Frac}\, P] = |W|$. Artin's fixed-field theorem [17, Tag 09I3] gives $[\mathrm{Frac}\, R : (\mathrm{Frac}\, R)^W] = |W|$ because the action is faithful. Moreover, $\mathrm{Frac}(R^W) = (\mathrm{Frac}\, R)^W$: multiply the denominator of an invariant rational function by its other group translates to make an invariant denominator, and then its numerator is invariant too. Since $P \subseteq R^W$, the equal field degrees imply $\mathrm{Frac}\, P = \mathrm{Frac}(R^W)$. Every element of R^W is integral over P, because R is finite over P, and lies in this common field. A polynomial ring is integrally closed, so $R^W = P$. □

In particular, after scalar extension, $\widetilde{C}$ is precisely $C = R^H/(R^W_+ R^H)$. Extension from L to $\mathbb{C}$ preserves ranks and dimensions in each finite graded piece. The specialization bound of Lemma 4.1 therefore applies to this characteristic-zero invariant quotient.

5 The free module and the Lefschetz comparison

We relate the auxiliary quotient to the relative algebra on which z will be Lefschetz. All rings in this section are over $\mathbb{C}$. In addition to C, put

$$A = R^K/(R^W_+ R^K), \quad F = R/(R^K_+ R),$$
$$B = R^H/(R^K_+ R^H), \quad E = R/(R^W_+ R).$$

Here F is the ordinary G_{33} coinvariant algebra; its fixed linear coordinate is killed in the quotient. The Hilbert series from the degrees in (2) give the following values. The top degree means the largest degree with a nonzero component.

Algebra	Dimension	Top degree
A	756	81
F	51 840	45
E	39 191 040	126
B	80	27
C	60 480	108

For completeness, each displayed relative quotient is a complete intersection in the appropriate polynomial invariant ring. It is finite because the smaller-group invariant ring is finite over the larger-group one. Its parameters therefore form a regular sequence, by the same Cohen–Macaulay argument as above. Taking the ratio of the products $1 - t^q$ gives

$$H_A(t) = h(t), \qquad H_B(t) = b(t), \qquad H_C(t) = h(t)b(t). \tag{8}$$

In particular A and F have symmetric Hilbert functions: replacing t by t^{-1} in their complete-intersection formulas reverses the coefficients about their respective top degrees.

Lemma 5.1. *There is a graded A-module isomorphism $C \cong A \otimes_{\mathbb{C}} B$ that intertwines multiplication by z. Consequently, for every $k \geq 1$,*

$$H_{C/(z^k)}(t) = H_{A/(z^k)}(t)b(t).$$

Proof. The reflection-coinvariant free-extension theorem [11, Proposition 3.1] gives $A \longrightarrow E$ with fiber F, since R^K is polynomial. The required injection also has a direct description: if a K-invariant belongs to the ideal $R_+^W R$, average its coefficients in a W-invariant generator expression over K. This proves $(R_+^W R) \cap R^K = R_+^W R^K$.

Choose a graded linear section $s : F \longrightarrow E$ of the quotient map. Averaging s over K makes it K-equivariant and leaves it a section. For every such section, the free-extension module isomorphism is

$$A \otimes_{\mathbb{C}} F \longrightarrow E, \qquad a \otimes f \longmapsto a\, s(f)$$

[11, Lemma 2.21]. This is an A-module isomorphism; no tensor-product algebra isomorphism is asserted. The action of K on A is trivial, so the displayed map is K-equivariant.

Taking H-invariants yields $E^H \cong A \otimes F^H$. Averaging over H is exact in characteristic zero. Applied to the two quotient maps, and averaging coefficients as above, it identifies $E^H = C$ and $F^H = B$. Normality of H in K is not needed. Since $z \in R^K$, the resulting isomorphism is z-linear. Quotienting by z^k proves the Hilbert-series identity. □

Proposition 5.2. *The class of z is a strong Lefschetz element of A.*

Proof. In degree j, the dimension of the cokernel of $z^k : A_{j-k} \to A_j$ is at least $\max(0, h_j - h_{j-k})$. Equality holds exactly when this multiplication has maximal rank. Here and below negative graded pieces are zero. Thus $H_{A/(z^k)} \geq Q_k$ coefficientwise. For $1 \leq k \leq 81$, Lemmas 4.1 and 5.1 and Proposition 3.1 give

$$Q_k b \;\leq\; H_{C/(z^k)} \;\leq\; H_{S/(I,z^k)} \;\leq\; H_{S/(J,z^k)} \;=\; Q_k b.$$

All inequalities are coefficientwise. Hence $(H_{A/(z^k)} - Q_k)b = 0$. The polynomial b has constant coefficient one and nonnegative coefficients. If the first factor had a first nonzero coefficient, the product would have that same nonzero coefficient in its first nonzero degree. Therefore $H_{A/(z^k)} = Q_k$, proving every required maximal rank for these k. For $k = 0$ multiplication is the identity. For $k > 81$, every target with nonzero source is zero because the top degree of A is 81. Degrees outside the indicated range require no additional check. □

Proof of Theorem 1.2. For G_{34} the algebra E is a graded Artinian free extension of A with fiber F. Proposition 5.2 gives SLP for A, and Wachi's result for G_{33} gives SLP for F [21, 22]. Both have symmetric Hilbert functions. Over a field of characteristic zero, SLP of the base and fiber therefore implies SLP of the extension [11, Proposition 2.22]; see also [8, Theorem 6.1]. These results allow nonstandard grading, and their Lefschetz elements lie in degree one. This proves the G_{34} case. The theorem supplies an element in E_1; it does not assert that z is Lefschetz on E or on C. All other irreducible types are covered by Table 1. □

Reducible groups and fixed directions

We now explain why the irreducible result also answers the original question for every finite complex reflection group.

Corollary 5.3. *For every finite complex reflection group W on a finite-dimensional complex vector space, the ordinary coinvariant algebra E_W has the strong Lefschetz property, including when the representation is reducible or has a nonzero fixed subspace.*

Proof. Average a positive Hermitian inner product over W and decompose orthogonally into the fixed subspace and nontrivial irreducible summands:

$$V = V^W \oplus V_1 \oplus \cdots \oplus V_r.$$

Each reflection preserves these summands. Since its difference from the identity has rank one, it acts nontrivially on exactly one summand. The subgroups generated by reflections supported on the respective V_i therefore commute, have disjoint supports, and generate W. Thus $W = W_1 \times \cdots \times W_r$, with W_i an irreducible reflection group on V_i and acting trivially on the other summands.

Put $R_i = \mathrm{Sym}(V_i^*)$. Expanding polynomials in the variables of the other summands shows coefficient by coefficient that invariance under W_i is precisely invariance of the R_i coefficients. Hence

$$R^W = \mathrm{Sym}((V^W)^*) \otimes_{\mathbb{C}} \bigotimes_{i=1}^{r} R_i^{W_i}.$$

Every fixed coordinate is a degree-one invariant and is killed in the coinvariant quotient. The remaining positive-degree invariant ideal is generated by the positive-degree invariants in each factor. Consequently there is a graded algebra isomorphism

$$E_W \cong \bigotimes_{i=1}^{r} R_i/((R_i^{W_i})_+ R_i) = \bigotimes_{i=1}^{r} E_{W_i}.$$

Each factor is graded Artinian and has SLP by Theorem 1.2. The Chevalley–Shephard–Todd polynomial invariant-ring theorem [18, Theorem 3] and the regular-sequence argument used above give

$$H_{E_{W_i}}(t) = \prod_j (1 + t + \cdots + t^{d_{ij}-1}),$$

where d_{ij} are the basic invariant degrees of W_i. In particular, each factor's Hilbert function is symmetric. Corollary 2.17 of [11] states that for graded Artinian algebras with symmetric Hilbert functions in characteristic zero, if x and y are degree-one strong Lefschetz elements of the factors, then $x \otimes 1 + 1 \otimes y$ is strong Lefschetz for their tensor product. All its hypotheses hold here. Tensor products preserve symmetry of the Hilbert function, so iteration proves SLP of E_W. If $r = 0$, the quotient is $\mathbb{C}$, and its zero degree-one element satisfies the definition directly. □

6 Certificates, reproduction, and trusted boundary

This section describes what the distributed programs establish and which parts of the theorem remain mathematical inputs or written arguments. The result is computer-assisted. There is no complete Lean proof of the theorem.

Certificate format

The mathematical certificate uses plain-text sparse polynomials in a, b, c, d, y, z over $\mathbb{F}_{1009}$. The first line is `CFG_POLY_V1`; the next line gives the prime, variable count, input count, and number of G_j. Records have the forms

`I i polynomial`	the six input polynomials,
`G j polynomial`	a homogeneous nonzero G_j,
`T i j polynomial`	a nonzero transformation entry.

Each polynomial is a sum of coefficient–monomial terms with explicit integer exponents and coefficients reduced modulo 1009. Transformation entries are grouped by increasing column, allowing streamed verification of $IT = G$. The file `data/unit-order.txt` specifies the prime, variable names, positive weights, and weighted degree reverse lexicographic order. The file `data/unit-initial.txt` gives candidate leading exponent tuples; the arithmetic checkers reconstruct them from the G_j instead of accepting them as an independent premise. The file `data/unit-starts.txt` gives candidate starts of the z-chains. The monomial checkers reconstruct the complete standard set and check the cover.

The additional pair certificate starts with `CFG_PAIRS_V1` and the number of basis polynomials. It treats every unordered pair exactly once, either by the product criterion, a chain criterion referring only to earlier accepted pairs, or a direct S-polynomial reduction. The dependency order rules out circular chain justifications. A second implementation audits the entire pair cover and repeats the direct arithmetic. This verifies Buchberger's criterion, although the proof above needs only the certified membership and resulting leading-ideal inclusion.

The arithmetic certificate has 534 membership columns. The pair cover has 142 311 pairs, divided into 1690 product pairs, 138 012 chain pairs, and 2609 direct reductions. The two monomial implementations both cover all 60 480 standard monomials and check all 81 quotients, including every degree and the finite tail. The packed-exponent implementation also checks the degree bounds that prevent exponent overflow.

Reproduction and independence

The accompanying README gives the requirements and one command that runs full verification from a clean checkout:

```
python3 checks/verify.py --workers 2
```

The six modular input files are in `data/inputs/`; the programs and their entry point are in `checks/`. The input and basis records are supplied in a compressed certificate file, together with the order, leading exponents, and chain starts. The much larger transformation records are reconstructed locally by the included Singular recipe before verification. The entry point combines the supplied input and basis records with these transformation records, then checks the resulting complete polynomial certificate. No external certificate download is needed.

The construction uses a Gröbner-basis computation to find a witness; its correctness is not a premise of acceptance. Neither arithmetic checker imports the producer or performs a basis search. Every coefficient identity is verified independently, so an incorrect reconstruction is rejected. The small supplied certificate records are not overwritten by the construction.

One arithmetic checker uses C++ with FLINT/GMP. The second uses a separate Python parser and Singular arithmetic, without calling a basis-search routine. Both independently bind the input records to the six modular polynomials and verify the coefficientwise identities. Separate integer implementations check the monomial cover and quotient Hilbert functions. The exact group model and the original-coordinate symbolic input comparisons are additional checks, not consequences of the Gröbner computation. Planted input, basis-coefficient, transformation-coefficient, leading-term, missing-pure-power, weight, and power-index errors are rejected by the corresponding checkers. These tests demonstrate rejection of the specified corruptions; they do not prove universal checker correctness. Independence here means distinct implementations, not independent authorship of the programs.

Literature and mathematical trust

The named group identifications, classification, and invariant degrees in (2) are imported from [14, 18, 13]. The exact order computation does not replace those premises. The earlier SLP cases, the tensor-product result, and the two free-extension results are imported at the locations cited above. In particular, the G_{33} fiber is not proved anew by these certificates.

The precise accessible source for Wachi's eight exceptional types is Wachi's author repository [22], whose introduction lists the proved cases and explicitly states that it has no proof for G_{34}. The abstract of the final article [21] states the all-except-one scope. The final article's proof and numbered theorem were not readable in the source review for this version. In particular, the final article's treatment of G_{34} remains unchecked; the 2024 author statement does not establish the scope of every later version. No final theorem number is asserted, and the author statement is not represented as a read proof of the final article. The theorem here imports that stated result, especially the G_{33} case. If its scope is corrected, the application above must be reconsidered.

The integral invariant presentation, specialization by relation matrices, invariant-generation argument, averaging, and Hilbert-function comparison, together with the product decomposition and removal of fixed coordinates, are proved in this paper. They are written mathematical arguments, outside a proof assistant. The exact execution trusts the parsers, Python integer arithmetic, SymPy's permutation-group algorithms, FLINT/GMP, Singular, and the C++ compiler and its runtime. No checker-soundness theorem or formal proof of the characteristic-zero bridges is claimed.

Further related work and credit

The invariant construction is that of Conway–Sloane; the subgroup model and convenient polynomial coordinates follow Oura–Sekiguchi. Their account also credits Burkhardt's earlier G_{33} invariants and the work of Orlik, Terao, Enta, Bessis, and Michel on exceptional invariants. The argu-

ment uses their structural information, with invariant generation also justified in Proposition 4.2. The transfer method owes its algebraic form to Harima–Watanabe and McDaniel–Chen–Iarrobino–Macias Marques. Wachi's exceptional computations and the earlier real, rank-two, and monomial results are indispensable to the classification-wide conclusion.

Watanabe's survey [23, §8] discusses the broader complete-intersection Lefschetz question and distinguishes it from known reflection cases. Smith's maximal-height work [16] addresses a related question; the joint McDaniel–Smith Bott–Samelson work, Smith's announced SLP deduction, and the McDaniel–Smith–Wachi breakout are discussed in the introduction. Maximal height and an ambient embedding alone do not imply the all-power ranks proved here. Kriebel's recent symmetric ideal-chain construction [9] concerns a different family and positive-characteristic behavior; no theorem or code from it is used. Nearby work includes Boij–Migliore–Miró-Roig–Nagel [2, Theorem 1.5] on weak Lefschetz maps in a specified degree range for equigenerated complete intersections, Nagaoka–Wachi [4, Theorem 4] on Gorenstein algebras generated by powers of relative invariants of regular prehomogeneous vector spaces of commutative parabolic type, and Gui–Sun–Wang–Zhu [3, Theorem 1.2] on the Kähler package for Bott–Samelson rings of Coxeter systems. Nothing from these three works is used here. This paper makes no claim of priority over unread later material.

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